Topics in Computational Social Choice 2026

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Approval Preferences

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Part 1
Approval Voting

The social choice pipeline

$3 \times \quad a \succ b \succ c$

$1 \times \quad a \succ c \succ b$

$1 \times \quad b \succ c \succ a$

$2 \times \quad c \succ a \succ b$

Preference profile

→ **Social choice function** →

$\{a\}$

Winner(s)

The social choice pipeline

$3 \times \quad \{a, b\}$

$1 \times \quad \{a, c\}$

$1 \times \quad \{b\}$

$2 \times \quad \{c\}$

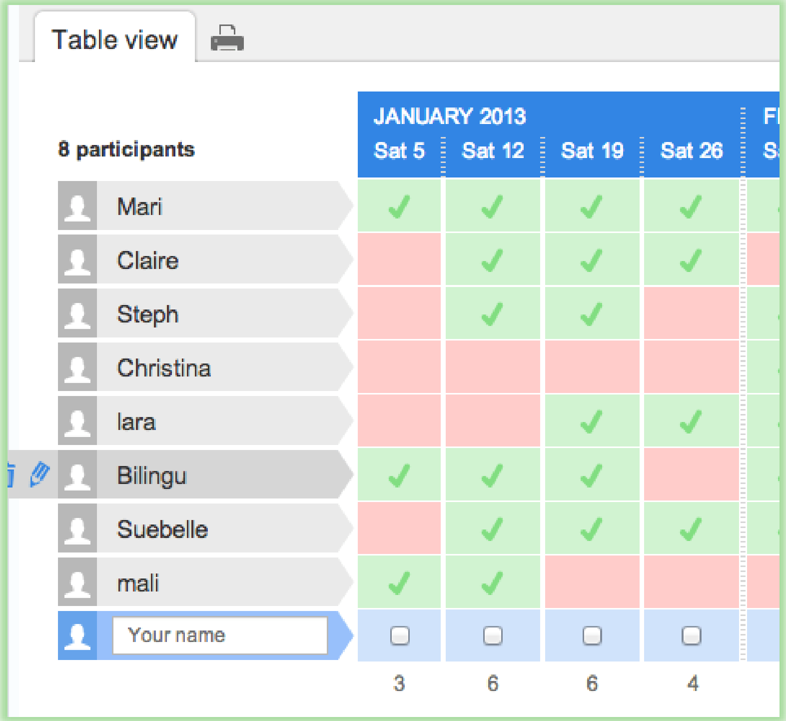
Approval profile

→ **Social choice function** →

$\{a\}$

Winner(s)

Examples of use cases



Selecting a date

Elections
(St. Louis, Fargo, ...)

Formal model of approval preferences

Fix a finite set $A = \{a, b, c, \dots\}$ of **alternatives** with $|A| = m \geq 2$.

An approval ballot is a subset of the preferences $B \subseteq A$. We denote by 2^A the set of all possible approval ballots.

Each **voter** of the finite set $N = \{1, \dots, n\}$ supplies an approval ballot B_i, giving rise to an **approval profile** $P = (B_1, \dots, B_n) \in (2^A)^n$.

An **approval-based voting** rule for A and N selects one or (in case of ties) more winners for every such profile:

$$F : (2^A)^n \to 2^A \setminus \{\emptyset\}$$

The approval voting rule

We define the **approval score** of an alternative x in a profile P as:

$$S_P(x) = |\{i \in N : x \in B_i\}|$$

The **approval rule** selects the alternatives with the highest approval score:

$$\mathrm{AV}(P) = \mathrm{argmax}_{x \in A} S_P(x)$$

Discussion: can you think of any other sensible approval-based voting rule?

Properties of approval voting

Anonymity: $F(B_1, \dots, B_n) = F\left(B_{\pi(1)}, \dots, B_{\pi(n)}\right)$ for any profile $P = (B_1, \dots, B_n)$ and permutation $\pi : N \to N$.
« All voters should be treated symmetrically »

Neutrality: $F\left(\pi(P)\right) = \pi\left(F(P)\right)$ for any profile P and permutation $\pi : A \to A$.
« All alternatives should be treated symmetrically »

Reinforcement:[1] For two profiles P on voter set N and P' on voter set N' and with the same alternative set A, we have $F(P + P') = F(P) \cap F(P')$ whenever $F(P) \cap F(P') \neq \emptyset$ where $P + P'$ is the concatenation of the two profiles.
« If an alternative wins in two voting stations, it should still win if we merge them »

[1] Sometimes called **Consistency**.

Arrovian properties

Let us now consider a **resolute** refinement of approval voting, for instance by **breaking ties lexicographically**.

Arrow's Theorem: « *Any **resolute** SCF for $m \geq 3$ alternatives that is **Paretian** and **independent** must be a **dictatorship**.* »

Arrow's impossibility theorem **do not apply** to approval voting.

Arrovian properties

Pareto: for a profile P and alternatives $x, y \in A$, if

(1) for every voter $i \in N$ we have $x \in B_i \Rightarrow y \in B_i$ and

(2) there is at least one voter $j \in N$ such that $y \in B_j$ and $x \notin B_j$

then $F(P) \neq x$.

« If every voter who approves x also approves y and at least one voter approves y and not x, then x should not be selected »

Independence: for two profiles $P = (B_1, \dots, B_n)$ et $P' = (B_1', \dots, B_n')$ and two alternatives $x, y \in A$, if for all $i \in N$, $B_i \cap \{x, y\} = B_i' \cap \{x, y\}$ then $F(P) = x \Rightarrow F(P') \neq y$.

« Whether x is socially preferred to y should depend only on whether x and y are approved in the profile (not on other, irrelevant, alternatives) »

Exercise: prove that approval voting satisfies these two properties.

Strategyproofness of approval voting

F is **strategy-proof** if for no voter $i \in N$ there exists a profile P (including i's truthful approval preference B_i) and an untruthful ballot B_i' for i such that $F(P_{-i}, B_i') \in B_i$ and $F(P) \notin B_i$.[2]

Exercise: Prove that approval voting is strategy-proof.

Remark: Gibbard-Satterthwaite Theorem does not apply to approval voting.

[2] **Notation:** (P_{-i}, B_i') is the profile obtained by replacing B_i by B_i' in P.

Part 2

Committee voting

Multi-winner (or committee) voting

$3 \times \quad \{a, b\}$

$1 \times \quad \{a, c\}$

$1 \times \quad \{b\}$

$2 \times \quad \{c\}$

Approval profile

→ **Social choice function** →

$\{\{b, c\}\}$

Committee(s)

Examples of use cases

Participatory Budgeting (PB)

Elections
(Society for Social Choice and Welfare)

Formal model of committee voting

We set a desired **committee size** $k \in \mathbb{N}$. A committee is a subset of A of size k. We denote $\binom{A}{k}$ the set of all committees of size k.

An **approval-based committee** rule (ABC rule) for A, N and k selects one or (in case of ties) more committees for every profile:

$$F : (2^A)^n \to 2^{\binom{A}{k}} \setminus \{\emptyset\}$$

Discussion: can you think of any sensible ABC rule?

Multi-winner approval voting

The simple **multi-winner approval voting rule** (MAV) selects the k alternatives with the highest score.

$$\mathrm{MAV}_k(P) = \mathrm{argmax}_{W \subseteq A, |W| = k} \sum_{x \in W} S_p(x)$$

What happens in the following profile, **with $k = 3$**?

$51 \times \{a, b, c\}$ $\quad$ $49 \times \{d, e, f\}$

Three main goals of committee voting

Excellence

We want the alternatives that are individually **the best**.

Use case: shortlisting, awards...

Rule: MAV

???

???

Three main goals of committee voting

Excellence

We want the alternatives that are individually **the best**.

Use case: shortlisting, awards…

Rule: MAV

Proportionality

We want to select alternatives such that each group is represented **in proportion to its size.**

Use case: assembly, participatory budgeting…

Rule: ??

Diversity

We want that as many voters as possible are represented by **at least one** alternative.

Use case: facility location…

Rule: ??

Chamberlain-Courant approval voting

The Chamberlain-Courant **approval voting rule** (CCAV) selects the k alternatives that cover the highest number of voters.

$$\text{CCAV}_k(P) = \text{argmax}_{W \subseteq A, |W|=k} |\{i \in N : W \cap B_i \neq \emptyset\}|$$

What alternatives are selected by MAV and CCAV in this profile, **with $k = 2$**?

$2 \times \{a\}$ $6 \times \{a, b\}$ $4 \times \{a, b, c\}$ $4 \times \{c, d\}$ $1 \times \{d\}$

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Proportional approval voting [Thiele, 1895]

Given a ballot B_i, **the PAV score** of a committee W of size k is equal to the sum of the $|B_i \cap W|$ first terms of the harmonic sequence $H = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

$$s_{PAV}(W, B_i) = \sum_{j=1}^{|B_i \cap W|} \frac{1}{j} = 1 + \cdots + \frac{1}{|B_i \cap W|}$$

The **proportional approval voting** (PAV) rule selects the committees maximizing the sum of the scores over all voters:

$$\text{PAV}_k(P) = \text{argmax}_{W \subseteq A, |W| = k} \sum_{i \in V} s_{PAV}(W, B_i)$$

Proportional approval voting

What alternatives are selected by PAV in this profile, **with $k = 2$**?

$2 \times \{a\}$ $6 \times \{a, b\}$ $4 \times \{a, b, c\}$ $4 \times \{c, d\}$ $1 \times \{d\}$

Proportional approval voting

What alternatives are selected by PAV in this profile, **with $k = 2$**?

$2 \times \{a\}$	$6 \times \{a, b\}$	$4 \times \{a, b, c\}$	$4 \times \{c, d\}$	$1 \times \{d\}$
1	$1 + 1/2$	$1 + 1/2$		
$2 \times \{a\}$	$6 \times \{a, b\}$	$4 \times \{a, b, c\}$	$4 \times \{c, d\}$	$1 \times \{d\}$
1	1	$1 + 1/2$	1	
$2 \times \{a\}$	$6 \times \{a, b\}$	$4 \times \{a, b, c\}$	$4 \times \{c, d\}$	$1 \times \{d\}$
1	1	1	1	1

Thiele rules

Given a ballot B_i, **the w-Thiele score** of a committee W of size k is equal to the sum of the $|B_i \cap W|$ first terms of the sequence $w = (w_1, w_2, w_3, \ldots)$

$$s_w(W, B_i) = \sum_{j=1}^{|B_i \cap W|} w_j$$

The **w-Thiele** rule selects the committees maximizing the sum of the scores over all voters:

$$w\text{−Thiele}_k(P) = \text{argmax}_{W \subseteq A, |W|=k} \sum_{i \in V} s_w(W, B_i)$$

Exercise: what is the vector w for MAV, CCAV and PAV?

Axiom: Committee monotonicity

We assume for now that the rule is **resolute** (assume fixed tie-breaking).

An ABC rule is **committee monotonic** if for a same profile P and some $k \in \mathbb{N}$, the committee selected with parameter k is a subset of the committee selected with parameter $k + 1$:

$$F_k(P) \subseteq F_{k+1}(P)$$

Exercise: show that MAV is committee monotonic.

Exercise: show that CCAV and PAV are not committee monotonic.

Sequential Thiele rules

Instead of maximizing the score of the committee as a whole, we will add alternatives to the committees one by one.

The **sequential w-Thiele** rule constructs the committee as follow:

- Start with the empty committee $\boldsymbol{W_0} = \emptyset$.
- For $k \geq 0$, the next alternative added to the committee is the one maximizing the contribution margin $\boldsymbol{W_{k+1}} = \boldsymbol{W_k} \cup \{\boldsymbol{x}\}$ with:

$$x = \mathrm{argmax}_{y \in A \setminus W_k} \sum_{i \in V} (s_w(W_k \cup \{y\}, B_i) - s_w(W_k, B_i))$$

Sequential **Thiele rules**

By construction, sequential Thiele rules **satisfy committee monotonicity**.

Moreover, it is possible to compute the results of a sequential Thiele rule in **polynomial time**, while Thiele rules (except MAV) are **NP-hard** to compute.

Three main goals of committee voting

Excellence

We want the alternatives that are individually **the best**.

Use case: shortlisting, awards...

Rule: MAV

Proportionality

We want to select alternatives such that each group is represented **in proportion to its size.**

Use case: assembly, participatory budgeting...

Rule: PAV, Seq-PAV, Phragmèn, Method of Equal Shares...

Diversity

We want that as many voter as possible is represented by **at least one** alternative.

Use case: facility location...

Rule: CCAV, Seq-CCAV

Measuring proportionality: apportionment

Informally, an outcome is **proportional** if groups of voters are represented in the committee proportionally to their size.

Consider first the case where voters approve **all candidates of (only) one party**.

This corresponds to the **apportionment** model.

Example: for the preferences below and a committee size $k = 4$, what should be the committee?

$$25 \times \{a_1, a_2, a_3, \dots\} \qquad 25 \times \{b_1, b_2, b_3, \dots\} \qquad 50 \times \{c_1, c_2, c_3, \dots\}$$

Measuring proportionality: apportionment

Informally, an outcome is **proportional** if groups of voters are represented in the committee proportionally to their size.

Consider first the case where voters approve **all candidates of one party**.

This corresponds to the **apportionment** model.

In this model, an outcome is proportional if a party supported by x voters **receives at least $\left\lfloor \frac{x}{n} \cdot k \right\rfloor$ seats in the committee.**

Question: how to generalize this principle outside of the apportionment model?

Axiom: Justified Representation

An outcome satisfies the minimal proportionality requirement if there is no **cohesive group of voters** of size n/k such that **all voters are unhappy with the committee**.

Axiom: Justified Representation

An outcome satisfies the minimal proportionality requirement if there is no **cohesive group of voters** of size n/k such that **all voters are unhappy with the committee**.

A group of voters $S \subseteq N$ is said to be **1-cohesive** if $|\cap_{i\in S} B_i| \geq 1$.

A committee W satisfies **Justified Representation (JR)** if for each group $S \subseteq N$ that is 1-cohesive and such that $|S| \geq n/k$, it holds that:

$$\left| W \cap \bigcup_{i\in S} B_i \right| \geq 1$$

In other words, at least one voter in S approve an alternative in W.

Axiom: Justified Representation

Example: we want a committee of size $k = 4$.

$3 \times \{a, b, c\}$ $3 \times \{b\}$ $3 \times \{e, f\}$ $2 \times \{x, y, z\}$ $1 \times \{x, y\}$

Axiom: Justified Representation

Example: we want a committee of size $k = 4$.

$3 \times \{a, b, c\}$ $3 \times \{b\}$ $3 \times \{e, f\}$ $2 \times \{x, y, z\}$ $1 \times \{x, y\}$

Axiom: Justified Representation

Example: we want a committee of size $k = 4$.

$3 \times \{a, b, c\}$ $3 \times \{b\}$ $3 \times \{e, f\}$ $2 \times \{x, y, z\}$ $1 \times \{x, y\}$

- The committee should contain a, b, or c.
- The committee should contain b.
- The committee should contain e or f.
- The committee should contain x, y, or z.

Axiom: Justified Representation

Example: we want a committee of size $k = 4$.

$3 \times \{a, b, c\}$ $3 \times \{b\}$ | $3 \times \{e, f\}$ | $2 \times \{x, y, z\}$ $1 \times \{x, y\}$

- ~~(The committee should contain a, b, or c.)~~
- The committee should contain b.
- The committee should contain e or f.
- The committee should contain x, y, or z.

Axiom: Justified Representation

An ABC rule satisfies **the Justified Representation axiom** if for every profile P, the committee it returns $f(P)$ satisfies Justified Representation.

Exercise: show that MAV fails JR.

Exercise: show that CCAV satisfies JR.

Extended Justified Representation

A group of voters $S \subseteq N$ is said to be **ℓ-cohesive** if $|\cap_{i \in S} B_i| \geq \ell$.

A committee W satisfies **Extended Justified Representation (EJR)** if for each group $S \subseteq N$ that is ℓ-cohesive and such that $|S| \geq \ell \cdot n/k$, it holds that:

$$|W \cap B_i| \geq \ell \qquad \text{for some voter } i \in S.$$

Theorem [Aziz et al., 2017]: PAV satisfies EJR and CCAV fails EJR.

The core

We say that a committee W is **in the core** if for each non-empty $N \subseteq V$ and each $\mathrm{T} \subseteq A$ with

$$\frac{|T|}{k} \leq \frac{|N|}{n}$$

there exists a voter $i \in N$ such that $|B_i \cap T| \leq |B_i \cap W|$, i.e., voter i is at least as satisfied with W as with T.

Question: is there always a committee in the core?

Other topics: variable size committee

We now assume that we can select any committee.

Use case: shortlisting

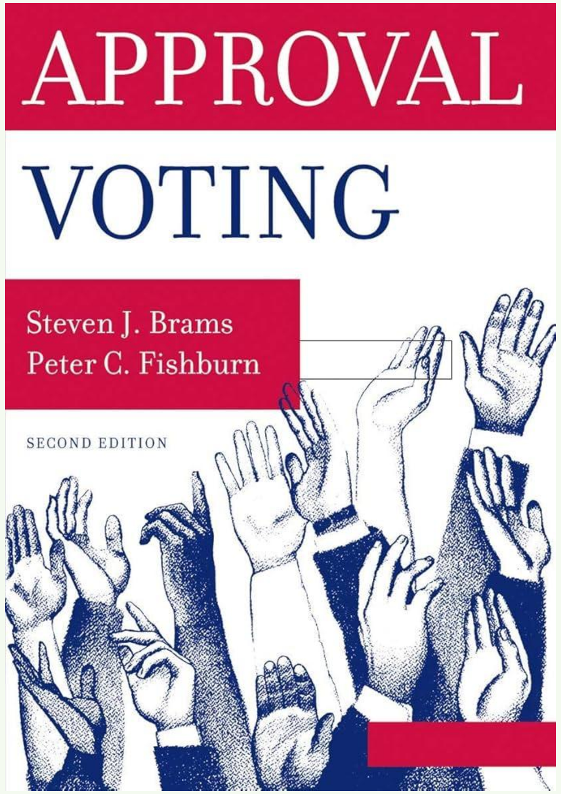
Question: what rule can we build specifically for this case?

Other topics: participatory budgeting

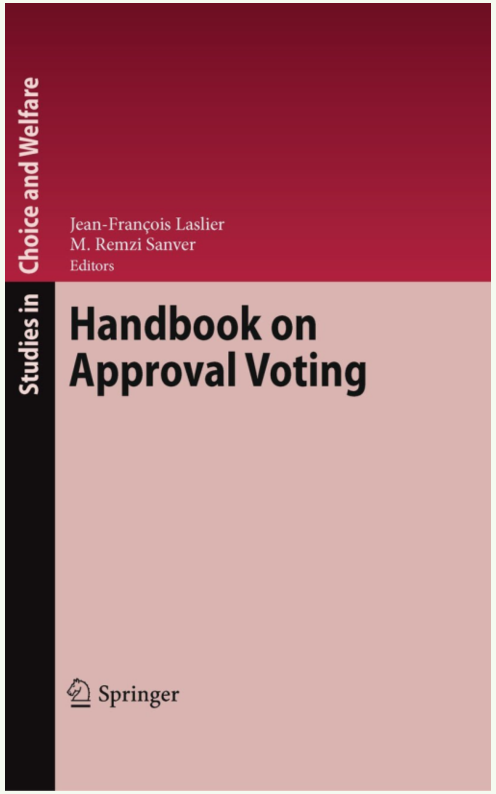
Each alternative is associated to a cost $cost(x) \geq 0$. The selected committee can be of any size but the total cost of the alternatives should not be higher than B.

Method of equal shares https://equalshares.net/explanation#example

Recommended books



1983



2010

2022

The social choice pipeline

Rankings
Approval Ballots
Ratings
...

→ **Social choice function** →

Winner(s)
Committee(s)
Assembly
Structure
...