# Approval-Compatible Voting Rules

## Positive Compatibility for Scoring Rules

Based on Terzopoulou–Lang–Zwicker

# Definitions

- Set of alternatives: $X$, $|X| = m \geq 3$.
- Voters: $N = \{1, \ldots, n\}$.
- Approval ballot: $A_i \subseteq X$, approval profile: $\boldsymbol{A} = \langle A_1, \ldots, A_m \rangle$
- Approval winners: App($\boldsymbol{A}$)
- rankings are strict total orders: $V_i = x_1 x_2 \ldots x_m$, ranking profile: $\boldsymbol{V} = \langle V_1, \ldots, V_n \rangle$

# Ranking Ballots and Compatibility

## Compatibility

A ranking profile $\boldsymbol{V}$ is compatible with approval profile $\boldsymbol{A}$, writing $\boldsymbol{V} \sim \boldsymbol{A}$, if

$$\forall A_i \in \boldsymbol{A}, \forall x, y \in X, (x \in A_i \land y \notin A_i) \implies x \succ_i y$$

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## Standardised ranking

Let $\boldsymbol{A}$ be a approval profile and let $x \in \text{App}(\boldsymbol{A})$. A ranking profile $\boldsymbol{V} \sim \boldsymbol{A}$ is a standardized ranking profile for $x$ under $\boldsymbol{A}$ if

1. $x \in A_i \implies x$ is ranked first in $V_i$
2. $x \notin A_i \implies x$ is ranked highest amongst nonapproved alternatives

# Formal Approval Compatibility Notions

## Positive Approval Compatibility (PAC)

A voting rule $r$ satisfies *PAC* if for every approval profile $\boldsymbol{A}$ and every $a \in \mathrm{App}(\boldsymbol{A})$, there exists a compatible ranking profile $\boldsymbol{V} \sim \boldsymbol{A}$ such that $a \in r(\boldsymbol{V})$

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**Obvious Positive Approval Compatibility (OPAC)**

$r$ satisfies *OPAC* if for every approval profile $\boldsymbol{A}$, every $a \in \mathrm{App}(\boldsymbol{A})$, and every standardised profile $\boldsymbol{V}$ for $a$ compatible with $\boldsymbol{A}$, $a \in r(\boldsymbol{V})$.

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## Fractional Positive Approval Compatibility (FPAC)

$r$ satisfies *FPAC* if for every approval profile $\boldsymbol{A}$ and approval winner $a$, there exists $k \geq 1$ and a compatible ranking profile $\boldsymbol{V} \sim k\boldsymbol{A}$ such that $a \in r(\boldsymbol{V})$.

# Positive Approval

**Uniform Positive Approval Compatibility (UPAC)**

$r$ satisfies *UPAC* if for every approval profile $\boldsymbol{A}$, there exists a compatible ranking profile $\boldsymbol{V} \sim \boldsymbol{A}$ such that $\mathrm{App}(\boldsymbol{A}) \subseteq r(\boldsymbol{V})$.

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**Fractional Uniform Positive Approval Compatibility (FUPAC)**

$r$ satisfies *FUPAC* if for every approval profile $\boldsymbol{A}$, there exists $k \geq 1$ and a compatible ranking profile $\boldsymbol{V} \sim k\boldsymbol{A}$ such that $\mathrm{App}(\boldsymbol{A}) \subseteq r(\boldsymbol{V})$.

# Theorem 1: Uniqueness of Plurality

**Theorem 1**

Plurality is the only positional scoring rule that satisfies OPAC.

**Proof.**

**Case 1**. Plurality.

- Consider an approval winner $x$.
- In a standardised profile:
  - Every voter approving $x$ ranks it first.
  - No alternative can get more first-place votes.
- Hence $x$ is a plurality winner.

□

# Proof of theorem 1

Reminder:

## Standardised ranking

Let $\boldsymbol{A}$ be a approval profile and let $x \in \text{App}(\boldsymbol{A})$. A ranking profile $\boldsymbol{V} \sim \boldsymbol{A}$ is a standardized ranking profile for $x$ under $\boldsymbol{A}$ if

1. $x \in A_i \implies x$ is ranked first in $V_i$
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## Proof.

**Case 2.** Different scoring rule $r$.

# Proof of theorem 1

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Let $\boldsymbol{A}$ be a approval profile and let $x \in \mathrm{App}(\boldsymbol{A})$. A ranking profile $\boldsymbol{V} \sim \boldsymbol{A}$ is a standardized ranking profile for $x$ under $\boldsymbol{A}$ if

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**Proof.**

**Case 2.** Different scoring rule $r$.

- Consider approval profile $\boldsymbol{A} = \langle \{x_1\}, \{x_2, \ldots, x_m\} \rangle$
- $x_1$ is an approval winner.
- Consider the standardised profile for $x_1 : \boldsymbol{V} = \langle x_1 x_2 \ldots x_m, x_2 x_3 \ldots x_m x_1 \rangle$
- $x_1$ does not win under $\boldsymbol{V}$, so $r$ does not satisfy OPAC.

□

# Proposition 2: Failure of $K$-Approval

**Lemma 2**

Let $r_s$ be the positional scoring rule with scoring vector $s = (s_1, \ldots, s_m)$. Then if $r_s$ is FPAC, the following holds $\forall l \in \{2, \ldots, m-1\}$

$$\frac{1}{m}(s_1 + \cdots + s_m) \leq \frac{l s_1 + (m-1)s_{l+1}}{l + m - 1}$$

# Proof of lemma 2

**Proof.**

- Suppose for contradiction it does not hold for some $l \in \{2, \dots, m-1\}$
- consider the profile $\boldsymbol{A}$ with $l + m - 1$ voters, where the first $l$ all only approve of $x_m$, and the remainign $m - 1$ circularly approve of the alternatives in $X \setminus \{x_m\}$
- Then all alternatives are approval winners
- Note that for every $\boldsymbol{V} \sim k\boldsymbol{A}$, the maximum score for $x_m$ is $k(ls_1 + (m-1)s_{l+1})$, and the sum of all scores is $k(l + m - 1)(s_1 + \cdots + s_m)$
- Thus, since our inequality is violated, $x_m$ receives a lower score then the average score, and thus will not be selected.

□

# Lemma 3: FUPAC condition

## Lemma 3 (Statement)

A positional scoring rule satisfying FUPAC must satisfy

$$\frac{s_1 + s_m}{2} = \frac{s_2 + \cdots + s_{m-1}}{m - 2}$$

# Theorem 3: Borda Satisfies FUPAC

Reminder:

**FUPAC**

$r$ satisfies *FUPAC* if for every approval profile $\boldsymbol{A}$, there exists $k \geq 1$ and a compatible ranking profile $\boldsymbol{V} \sim k\boldsymbol{A}$ such that $\mathrm{App}(\boldsymbol{A}) \subseteq r(\boldsymbol{V})$.

**Theorem 3**

The Borda rule satisfies FUPAC.

# Proof of Theorem 3

**Proof.**

- Fix some labeling of the alternatives $x_1, \ldots, x_n$
- For the approval profile $\boldsymbol{A}$, consider the ranking profile $\boldsymbol{V} = \langle \boldsymbol{V}^1, \boldsymbol{V}^2 \rangle \sim 2\boldsymbol{A}$ , where
  - $V_i^1 \in \boldsymbol{V}^1$ contains the lexicographic ordering of approved alternatives at the top, and lexicographic ordering of the rest under it
  - $V_i^2 \in \boldsymbol{V}^2$ contains the reverse lexicographic ordering of approved alternatives at the top, and the reverse lexicographic ordering of the rest under it
- Let $m(x, y) = |N_{x \succ y}| - |N_{y \succ x}|$. Then $\sum_{y \in X} m(x, y) = \beta(x) =$ symmetric Borda rule.
- $m(x, y) = 2(\text{App}_{\boldsymbol{A}}(x) - \text{App}_{\boldsymbol{A}}(y))$, from which follows $\beta(x) = \sum_{y \in X} 2(\text{App}_{\boldsymbol{A}}(x) - \text{App}_{\boldsymbol{A}}(y)) = 2m\text{App}_{\boldsymbol{A}}(x) - 2\sum_{y \in X} \text{App}(y)$

□

# Proposition 3: UPAC is Too Strong

Reminder:

**UPAC**

$r$ satisfies *UPAC* if for every approval profile $\boldsymbol{A}$, there exists a compatible ranking profile $\boldsymbol{V} \sim \boldsymbol{A}$ such that $\mathrm{App}(\boldsymbol{A}) \subseteq r(\boldsymbol{V})$.

**Proposition 3**

None of the considered rules are UPAC.

**Proof.**

Take 1 voter that approves all canditates. □

# Theorem 2

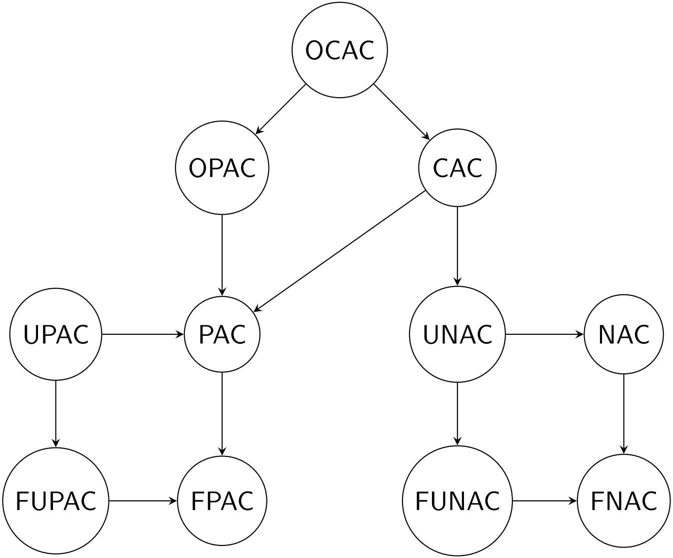
**Theorem 2**

The Borda rule satisfies PAC.

# Approval Compatibility Results

| | OPAC | FPAC | PAC | UPAC | FUPAC |
|---|---|---|---|---|---|
| Plurality | ✓ | ✓ | ✓ | × | × |
| $K$-approval ($K \geq 2$) | × | × | × | × | × |
| Borda | × | ✓ | ✓ | × | ✓ |

# Logical Relations Between Compatibility Notions



# Approval Compatibility: Full Classification

| | OCAC | OPAC | CAC | PAC |
|---|---|---|---|---|
| Plurality | ✓ | ✓ | ✓ | ✓ |
| $K$-approval ($K \geq 2$) | × | × | × | × |
| Borda | × | × | ✓ | ✓ |
| STV | ✓ | ✓ | ✓ | ✓ |
| Plurality runoff | ✓ | ✓ | ✓ | ✓ |
| Condorcet-consistent | ✓ | ✓ | ✓ | ✓ |

| | FPAC | UPAC | FUPAC | NAC | FNAC | UNAC | FUNAC |
|---|---|---|---|---|---|---|---|
| Plurality | ✓ | × | × | ✓ | ✓ | ✓ | ✓ |
| $K$-approval ($K \geq 2$) | × | × | × | ✓ | ✓ | × | × |
| Borda | ✓ | × | ✓ | ✓ | ✓ | ✓ | ✓ |
| STV | ✓ | × | × | ✓ | ✓ | ✓ | ✓ |
| Plurality runoff | ✓ | × | × | ✓ | ✓ | ✓ | ✓ |
| Condorcet-consistent | ✓ | × | ✓ | ✓ | ✓ | ✓ | ✓ |

# Refferences

Zoi Terzopoulou, Jérôme Lang, William Zwicker. Approval compatible voting rules. Social Choice and Welfare, 2025, ff10.1007/s00355-025-01588-9ff. ffhal-05393305f