Notions of Single-Peakedness for Incomplete Preferences

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Context: Voting and preferences

We know the preference **rankings** of several **voters** over a set of **candidates**.

$$C \succ B \succ D \succ A \succ E$$

$$D \succ C \succ E \succ B \succ A$$

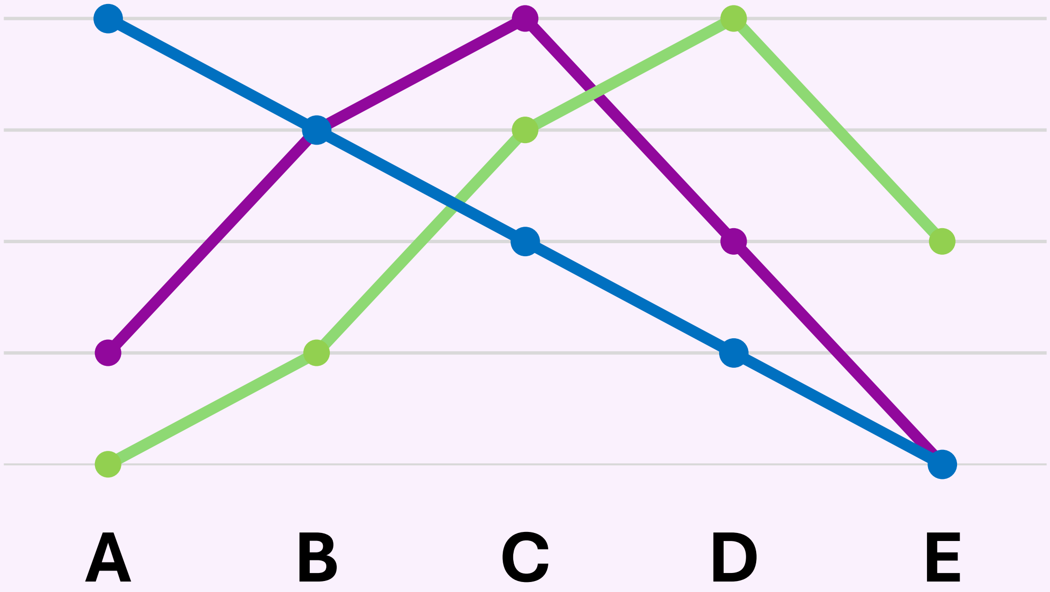
$$A \succ B \succ C \succ D \succ E$$

$$D \succ C \succ E \succ B \succ A$$

…

Single-peakedness for total orders

Preferences are **single-peaked** if there is an **axis** such that for each voter, their preference decreases the further we get from the peak.



$C \succ B \succ D \succ A \succ E$

$D \succ C \succ E \succ B \succ A$

$A \succ B \succ C \succ D \succ E$

Single-peakedness for total orders

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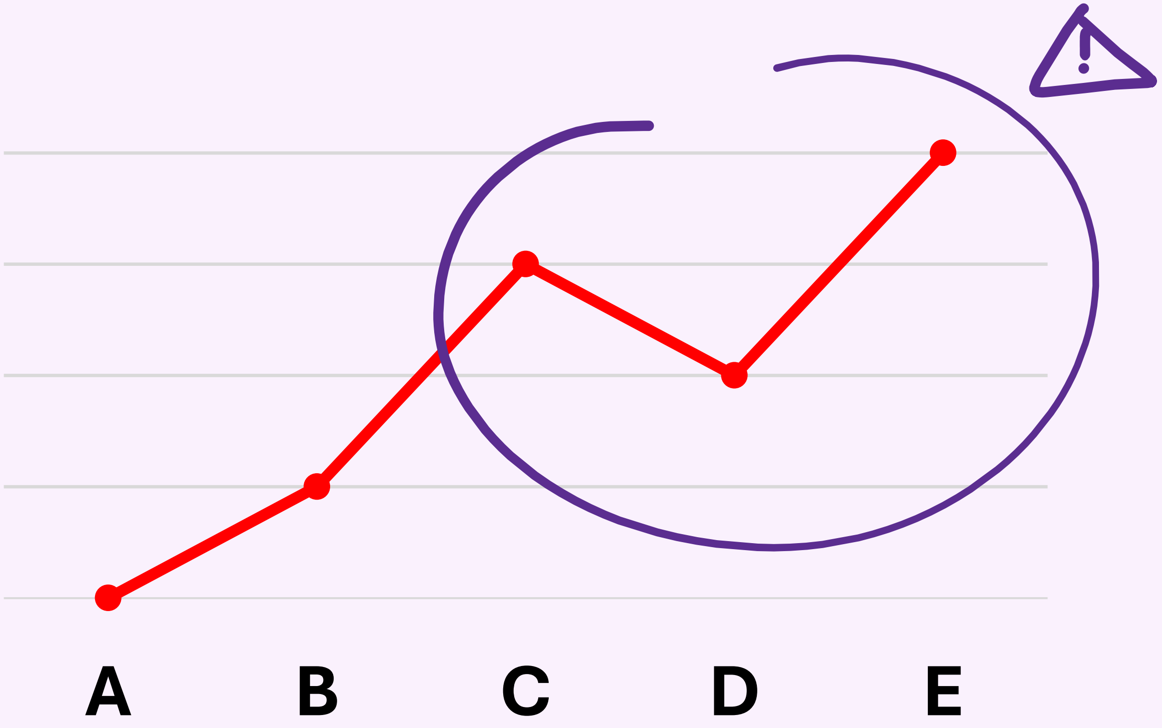


$E \succ C \succ D \succ B \succ A$

What is nice about single-peaked preferences?

1 Preference analysis

It highlights a ***meta-consensus*** on a unifying dimension for the candidates.

2 Decision making

Impossibility theorems do not apply: **there is a Condorcet winner** and the pairwise majority relation is transitive.

Single-peakedness in deliberative polls

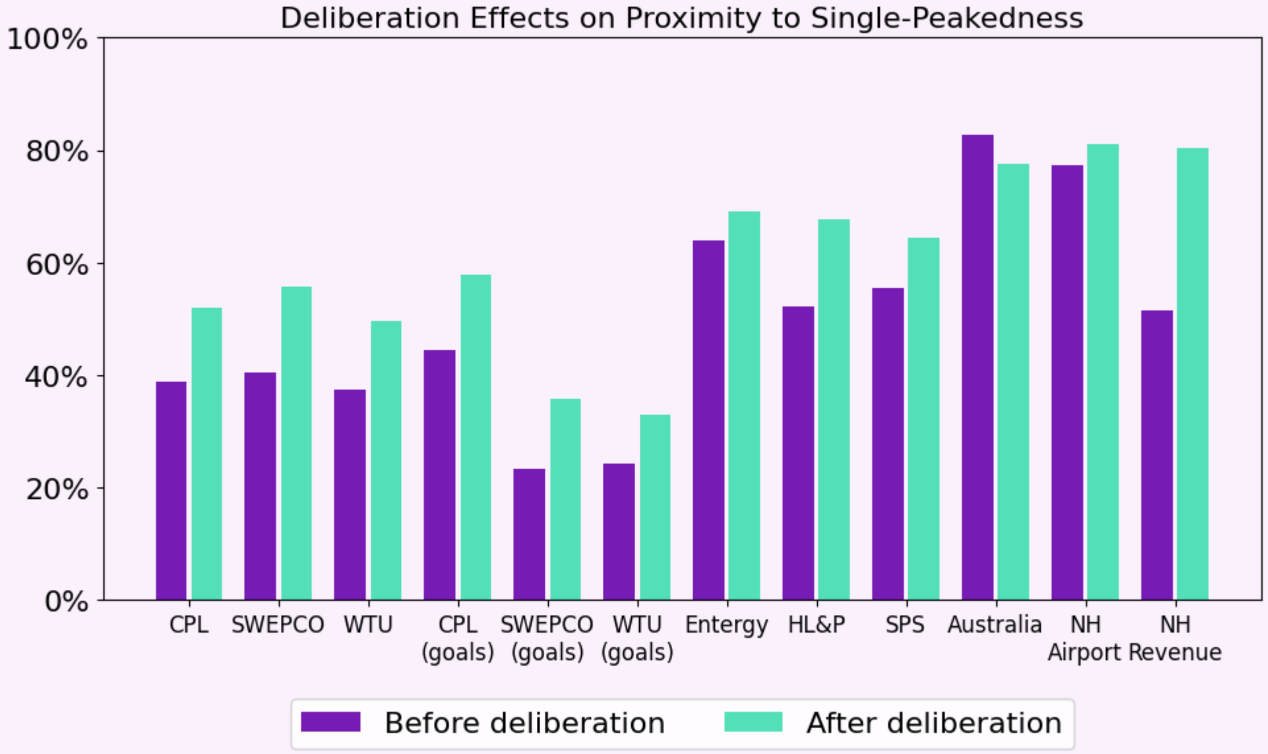
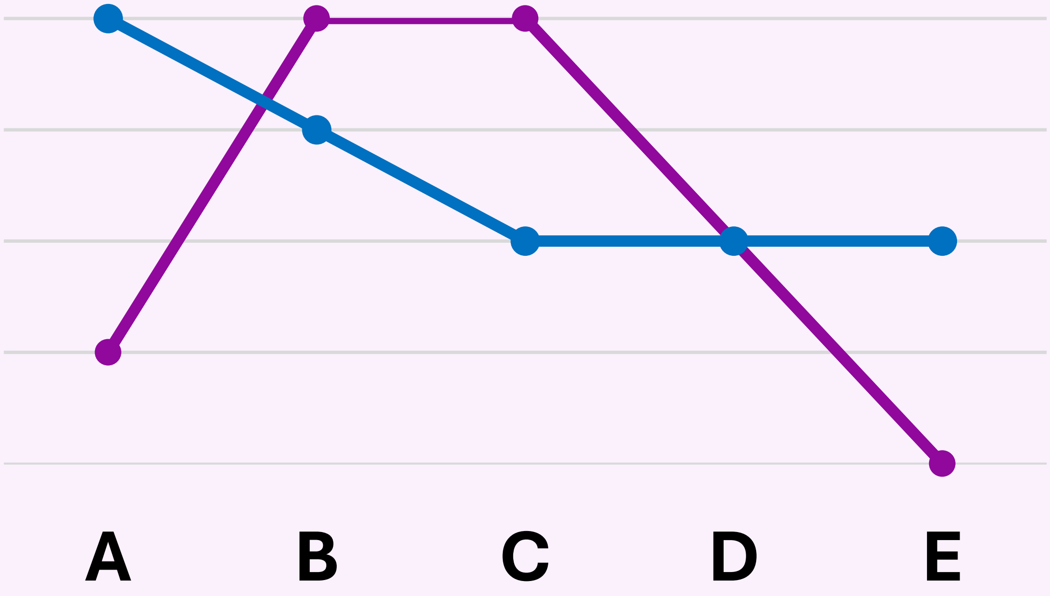


Fig. People in a Deliberative Polls
(Source: Deliberative Democracy Lab)

Deliberation, Single-Peakedness, and the Possibility of Meaningful Democracy: Evidence from Deliberative Polls. List et al. **The Journal of Politics (2013)**

What if we have indifferences?

Sometimes voters (can) express **indifferences** between some candidates in their ranking, putting them at the same rank.



$B \sim C \succ D \succ A \succ E$

$A \succ B \succ C \sim D \sim E$

Question: *Are these preferences single-peaked?*

What if we have incomplete preferences?

Sometimes voters (can) provide **incomplete** preferences, meaning we do not know their preferences between some candidates.

$C \succ B \succ D \succ A$

$B \succ C \succ A$

$D \succ E$

Question: *Are these preferences single-peaked?*

Question: when preferences contain **indifferences** and **incompleteness**, how should we define single-peakedness?

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Ten single-peakedness notions

Model: Incomplete preferences

We have a set of **voters** $\{1, \ldots, n\}$ and a set of **candidates** $\{A, B, C, \ldots\}$.

Voters give **preferences** over the candidates:

- **Strict preference:** $\boldsymbol{A} \succ \boldsymbol{B}$
- **Indifference:** $\boldsymbol{A} \sim \boldsymbol{B}$

} $\boldsymbol{A} \succcurlyeq \boldsymbol{B}$ (weak preference)

Preferences must be **transitive**, but not necessarily **complete**:

$A \succ C \succ D \succ B$

(Total order)

$A \sim C \succ D \succ B$

(Weak order)

$A \sim C \succ D$

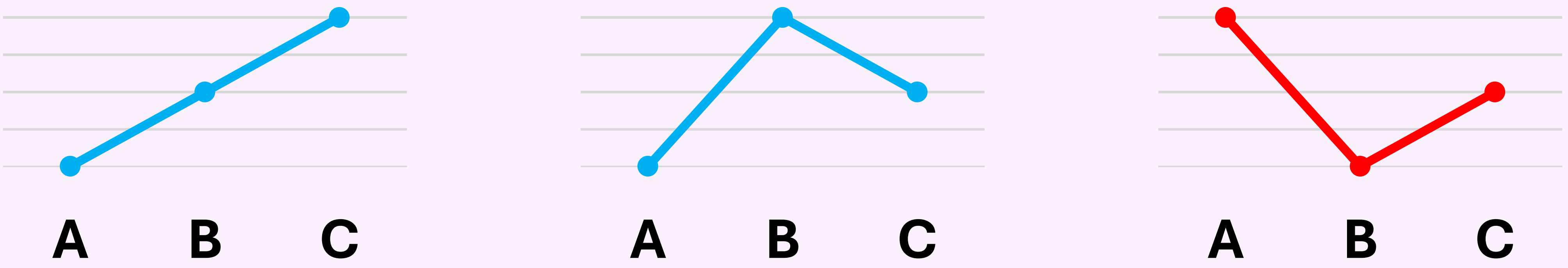
(Incomplete order)

Formal definition of single-peakedness

With **total orders**, the condition for single-peakedness is as follows:

If B is between A and C on the "**axis**", then:

$$B \succ_i A \text{ OR } B \succ_i C$$

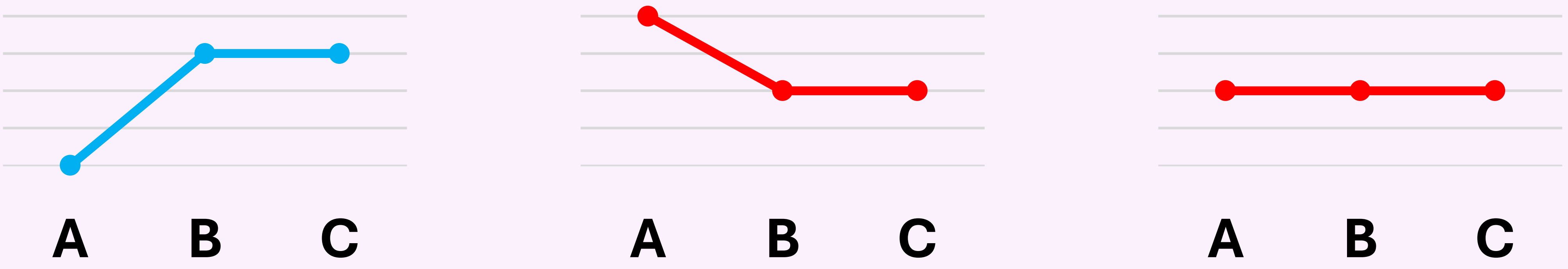


Formal definition of single-peakedness

If **B** is between **A** and **C** on the “axis”, then:

$$B \succ_i A \text{ OR } B \succ_i C$$

But if we apply it to rankings with **indifferences**:

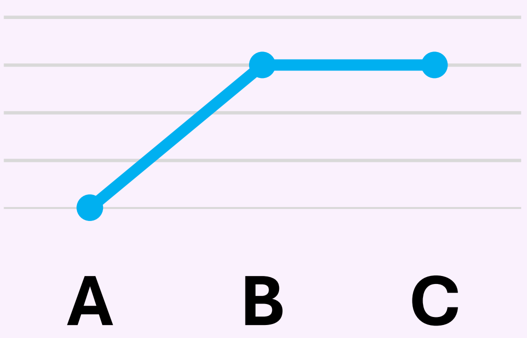


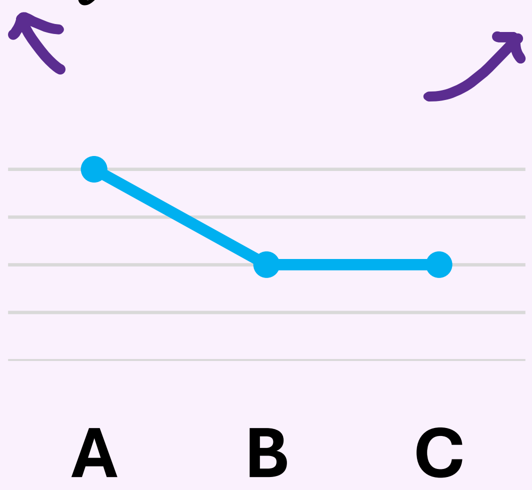
Formal definition of single-peakedness

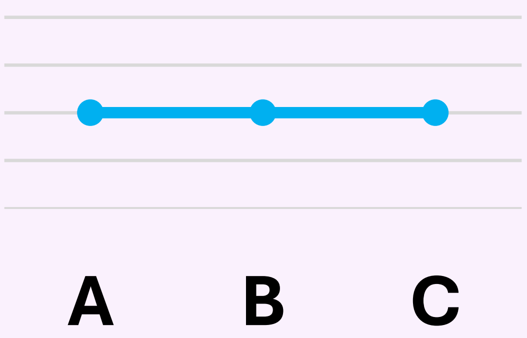
We could slightly adapt it:

If B is between A and C on the "axis", then:

$$B \succcurlyeq_i A \text{ OR } B \succcurlyeq_i C$$







Formal definition of single-peakedness

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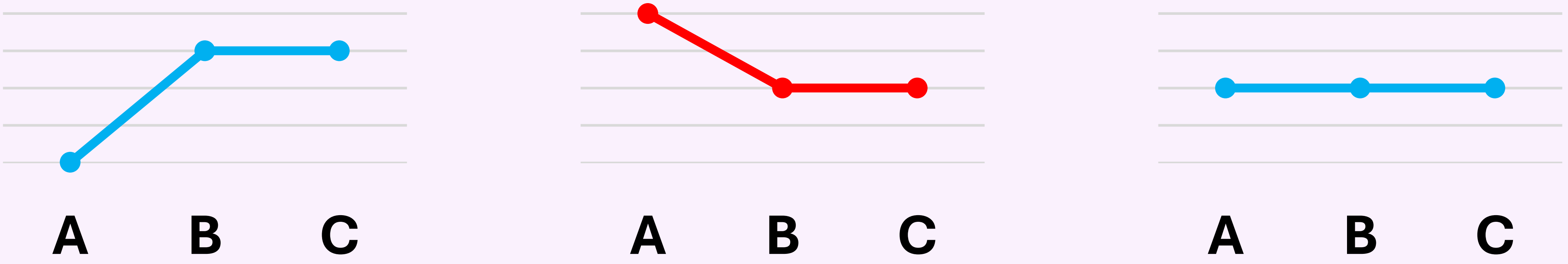


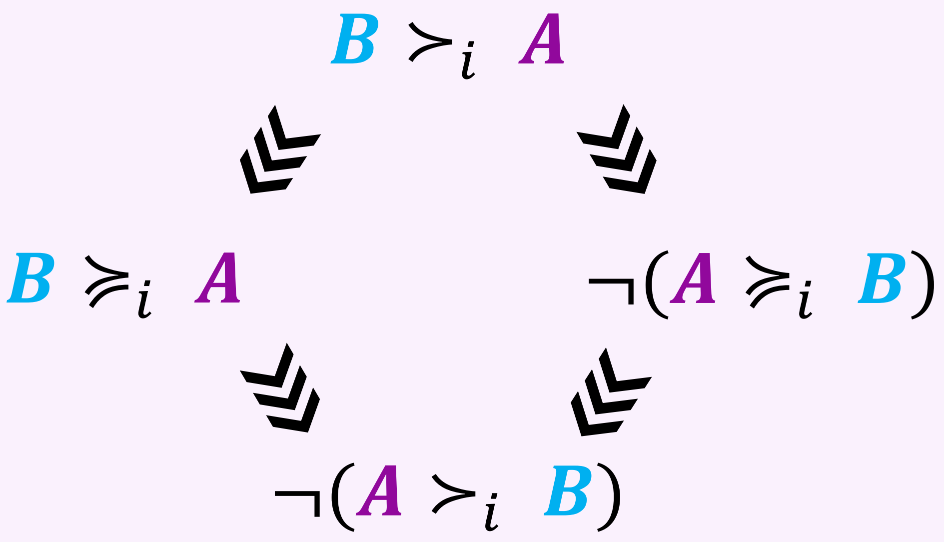
Let's now add incomparability

	$\succ$	$\sim$	?
$B \succ_i A$	✓		
$B \succcurlyeq_i A$	✓	✓	
$\neg(A \succcurlyeq_i B)$	✓		✓
$\neg(A \succ_i B)$	✓	✓	✓

Let's now add incomparability

	$\succ$	$\sim$	$?$
$B \succ_i A$	✓		
$B \succcurlyeq_i A$	✓	✓	
$\neg(A \succcurlyeq_i B)$	✓		✓
$\neg(A \succ_i B)$	✓	✓	✓

Hierarchy of strength



All combinations

$$\begin{array}{c} B \succ_i A \\ B \succcurlyeq_i A \\ \neg(A \succcurlyeq_i B) \\ \neg(A \succ_i B) \end{array} \quad \times \quad \begin{array}{c} B \succ_i C \\ B \succcurlyeq_i C \\ \neg(C \succcurlyeq_i B) \\ \neg(C \succ_i B) \end{array}$$

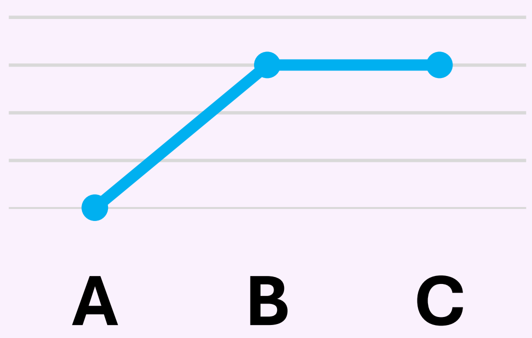
All combinations

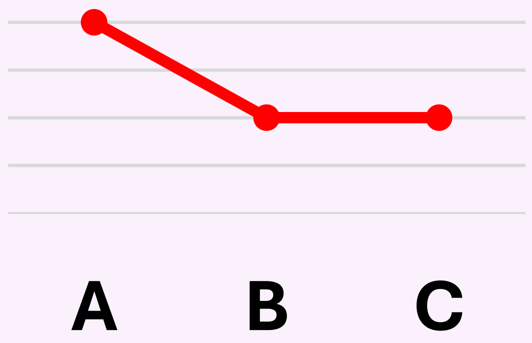
$$B \succ_i A$$
$$B \succcurlyeq_i A$$
$$\neg(A \succcurlyeq_i B)$$
$$\neg(A \succ_i B)$$

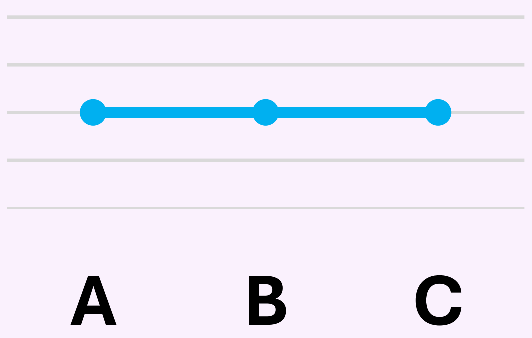
$$\times$$

$$B \succ_i C$$
$$B \succcurlyeq_i C$$
$$\neg(C \succcurlyeq_i B)$$
$$\neg(C \succ_i B)$$

$$B \succcurlyeq_i A \text{ OR } B \succ_i C$$







A hierarchy of ten notions

$B \succ_i A \text{ OR } B \succ_i C$

$B \succ_i A \text{ OR } B \succcurlyeq_i C$

$B \succ_i A \text{ OR } \neg(C \succcurlyeq_i B)$

$B \succcurlyeq_i A \text{ OR } \neg(C \succcurlyeq_i B)$

$B \succcurlyeq_i A \text{ OR } B \succcurlyeq_i C$

$\neg(A \succcurlyeq_i B) \text{ OR } \neg(C \succcurlyeq_i B)$

$B \succ_i A \text{ OR } \neg(C \succ_i B)$

$B \succcurlyeq_i A \text{ OR } \neg(C \succ_i B)$

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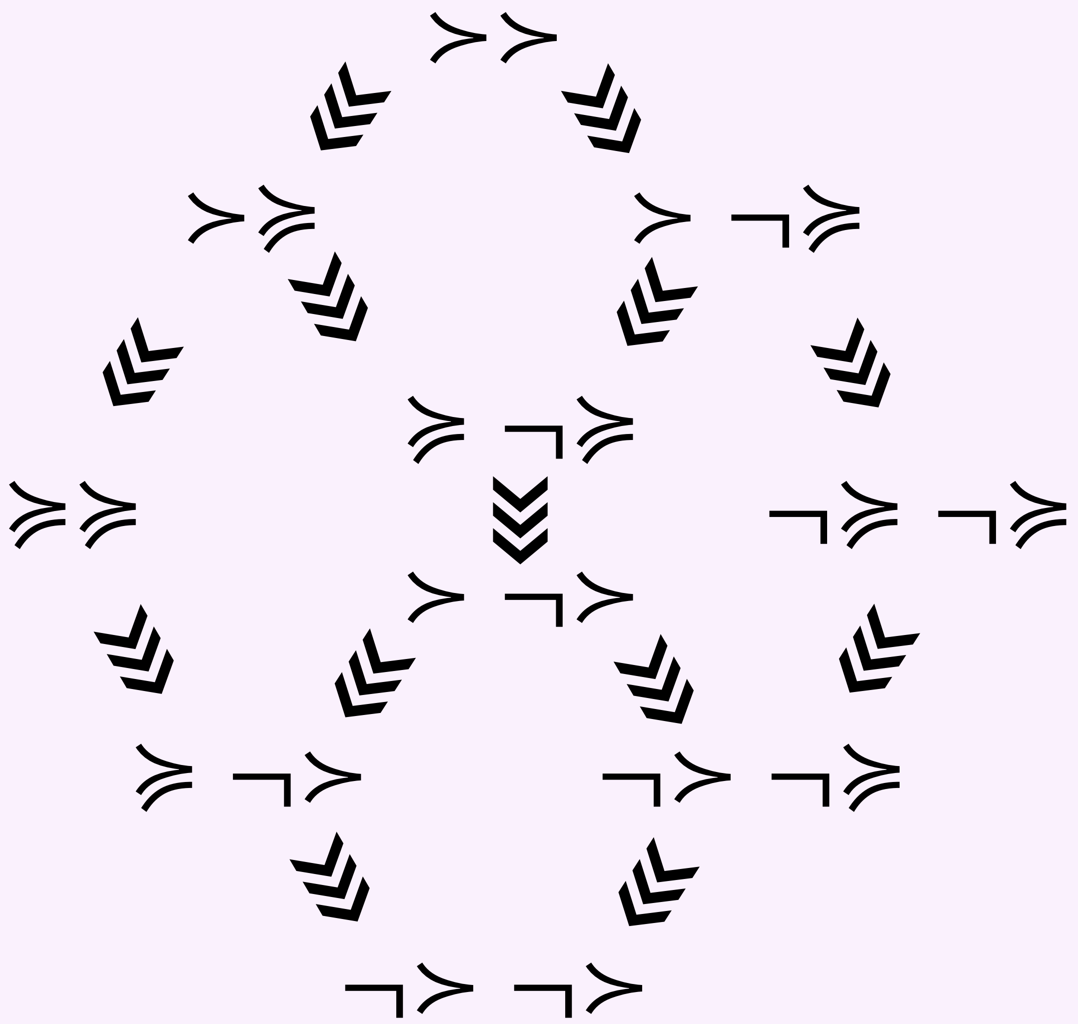

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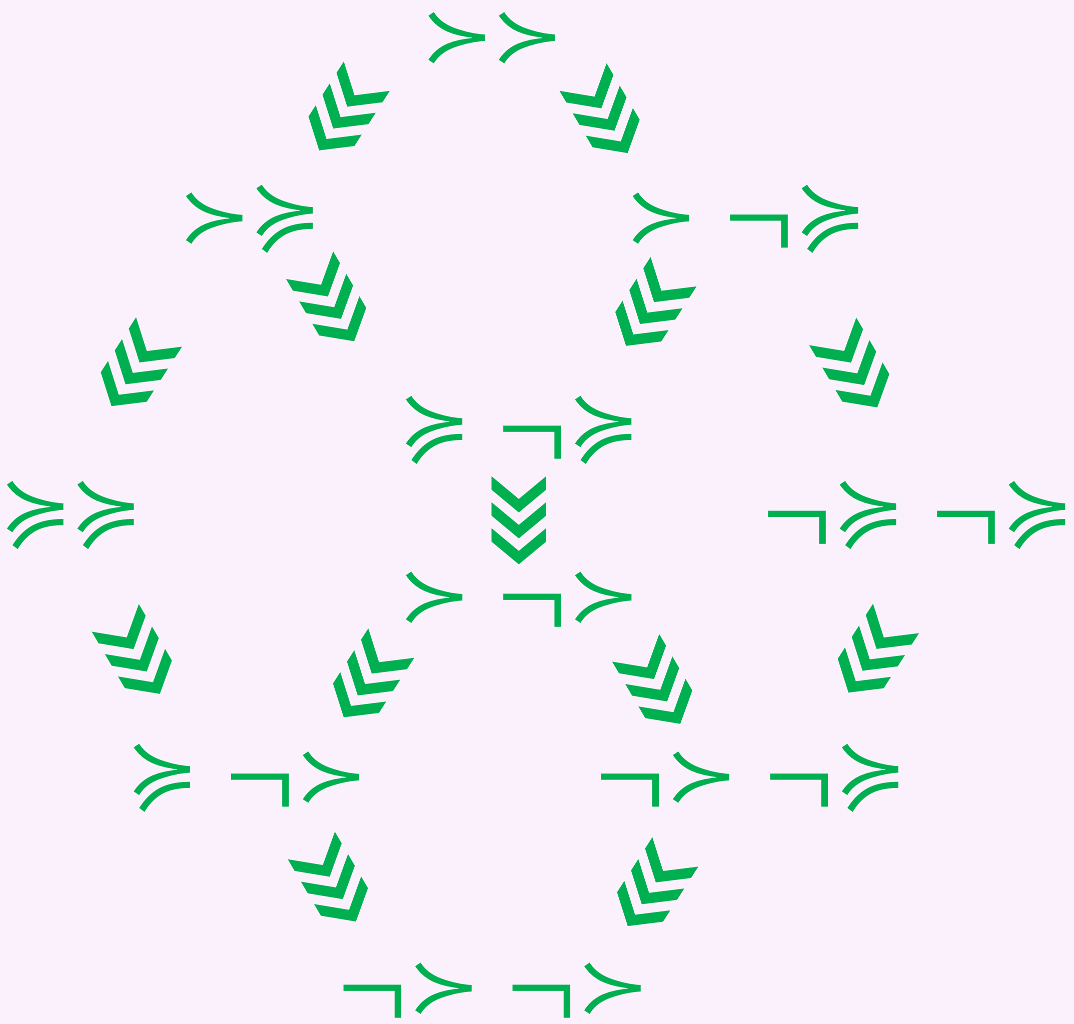
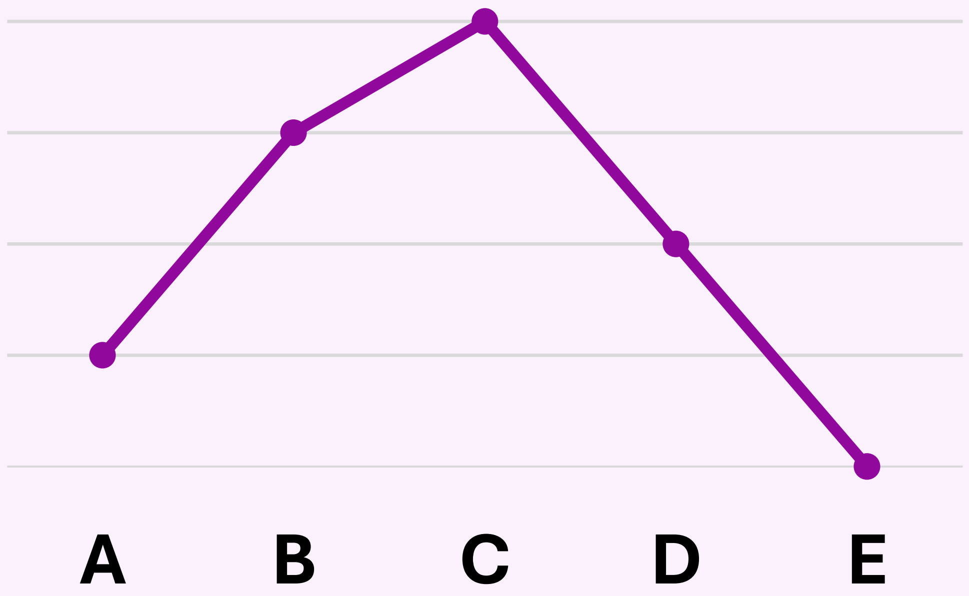


Is this single-peaked?



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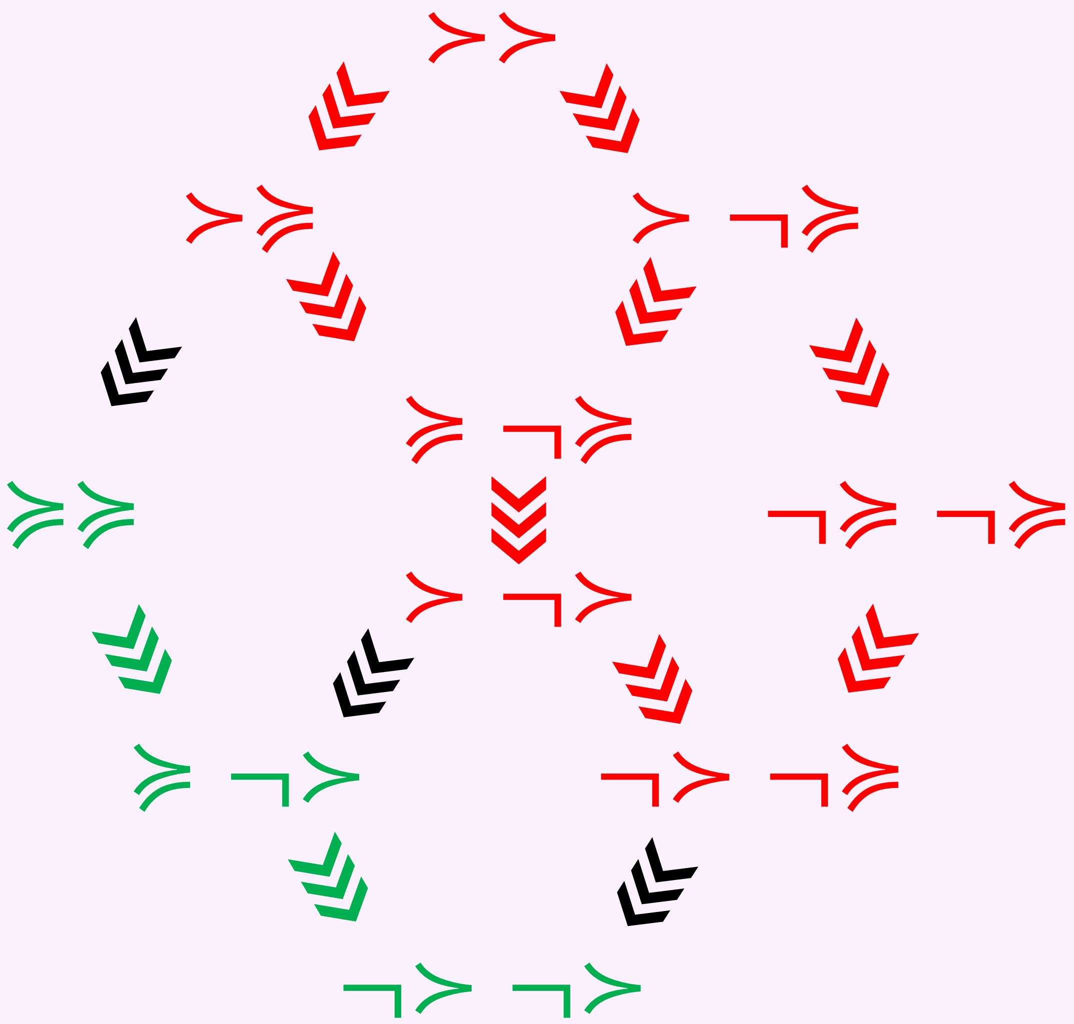

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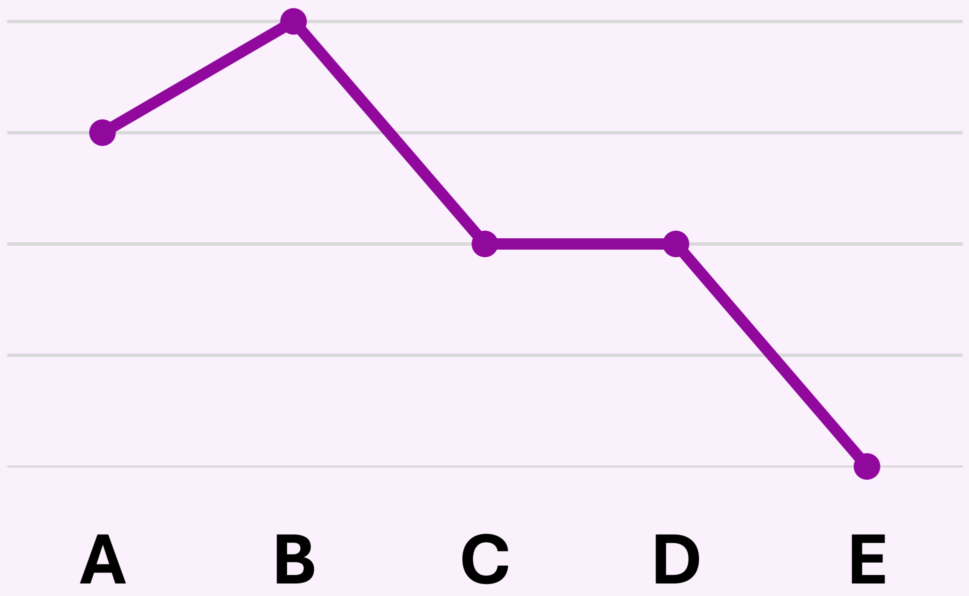

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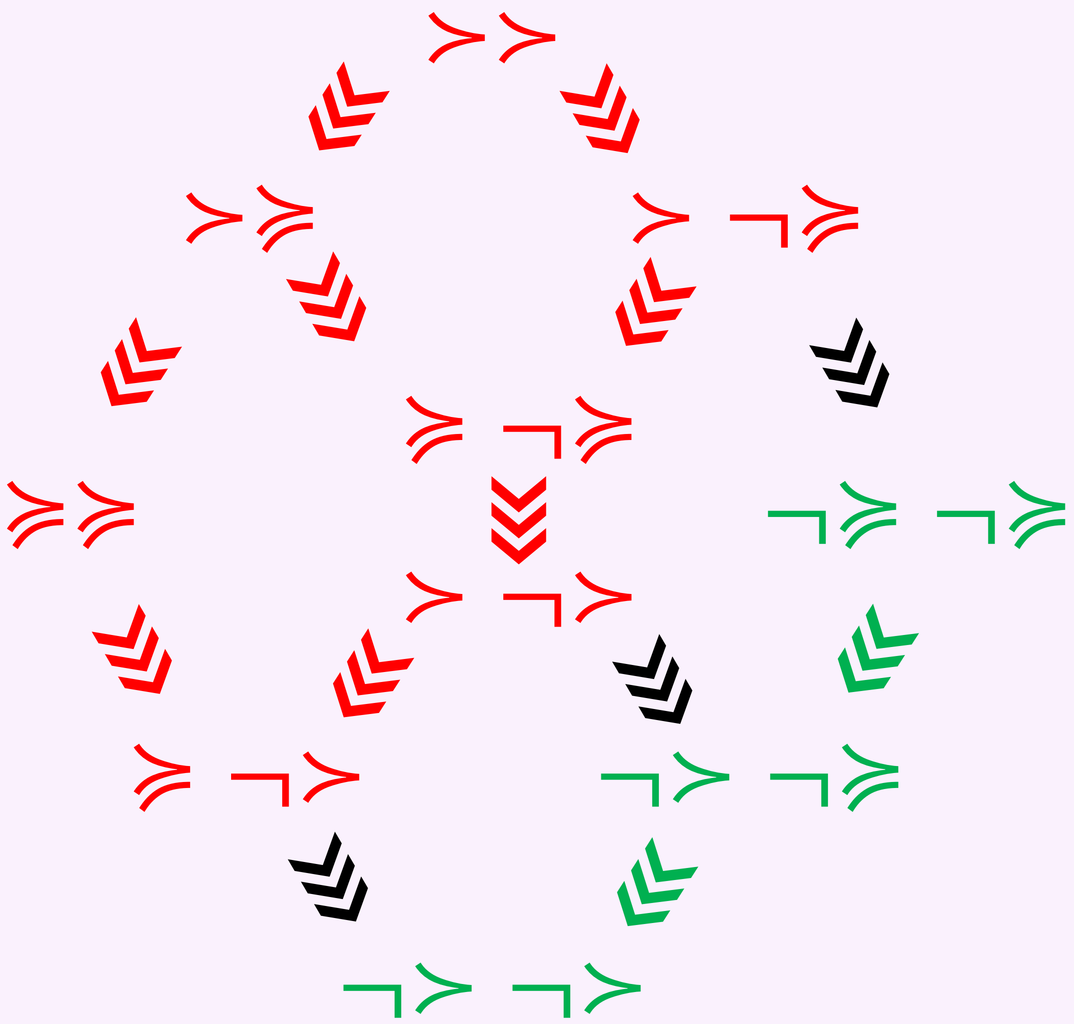
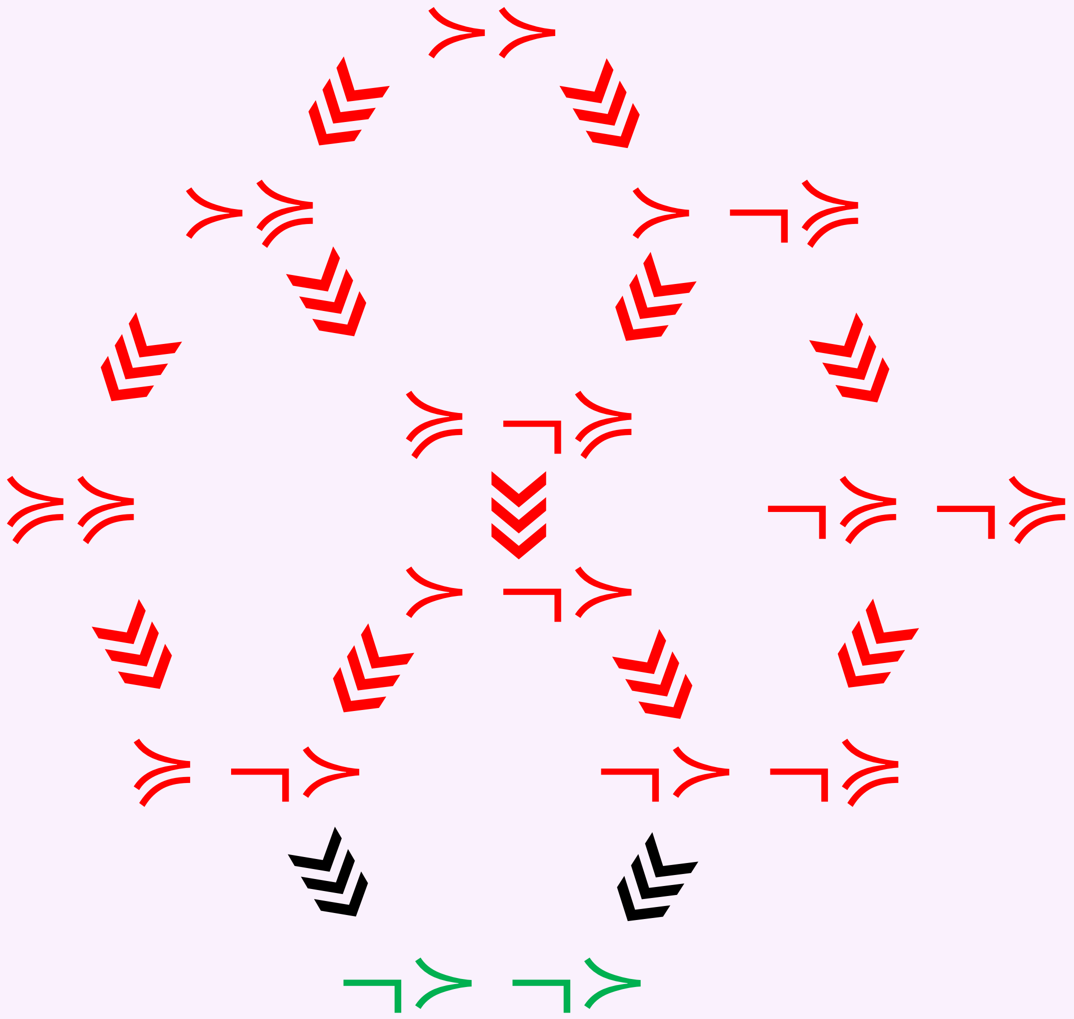


Is this single-peaked?

A hierarchy of ten notions



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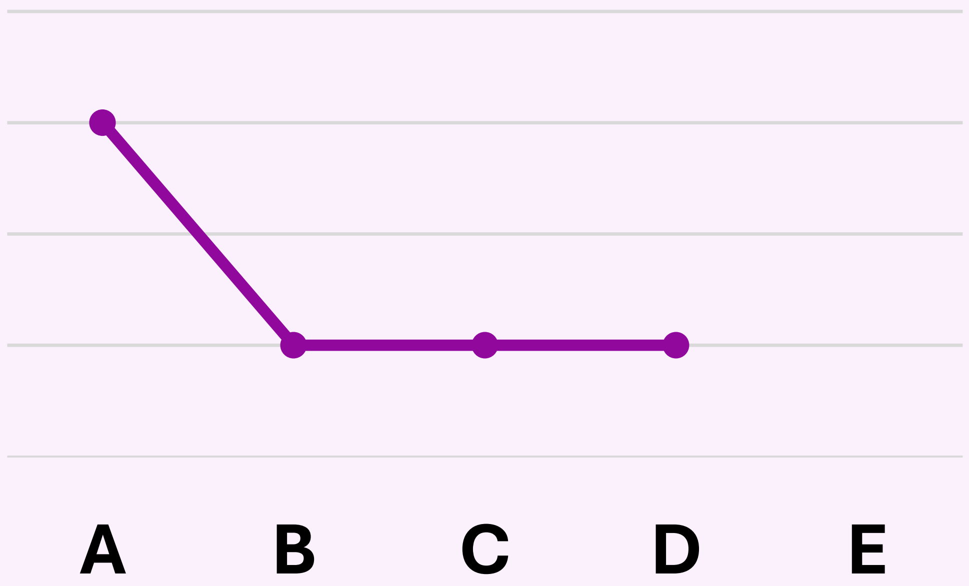


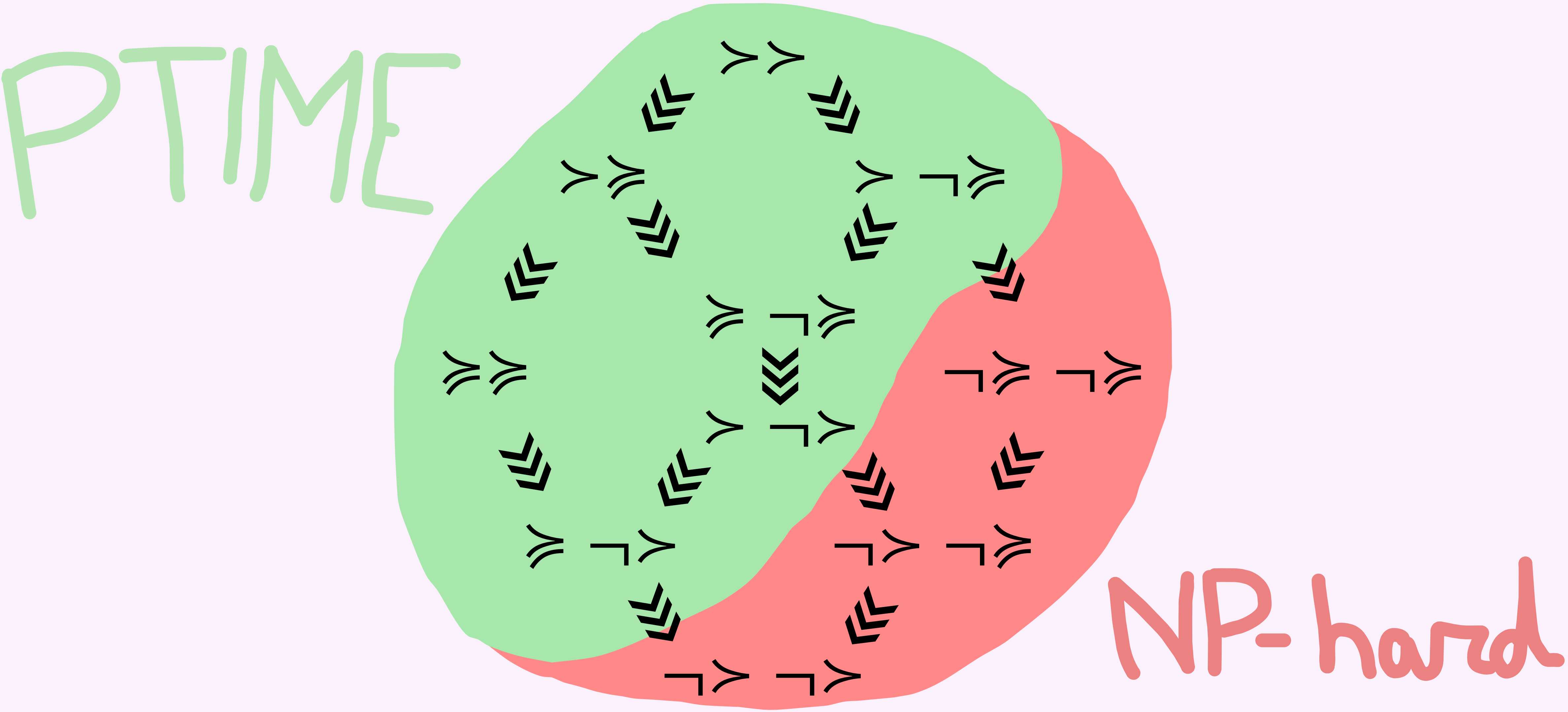
- 2 -
Theoretical Analysis of the Different Notions

Complexity of verifying single-peakedness

Question: how complex is it to compute whether a profile of preferences satisfies the single-peakedness condition?

- **PTIME:** We **can** compute it easily, which makes it practical to use.
- **NP-Hard:** We **cannot** compute easily when the number of candidates is large, making it unpractical.

Complexity of verifying single-peakedness



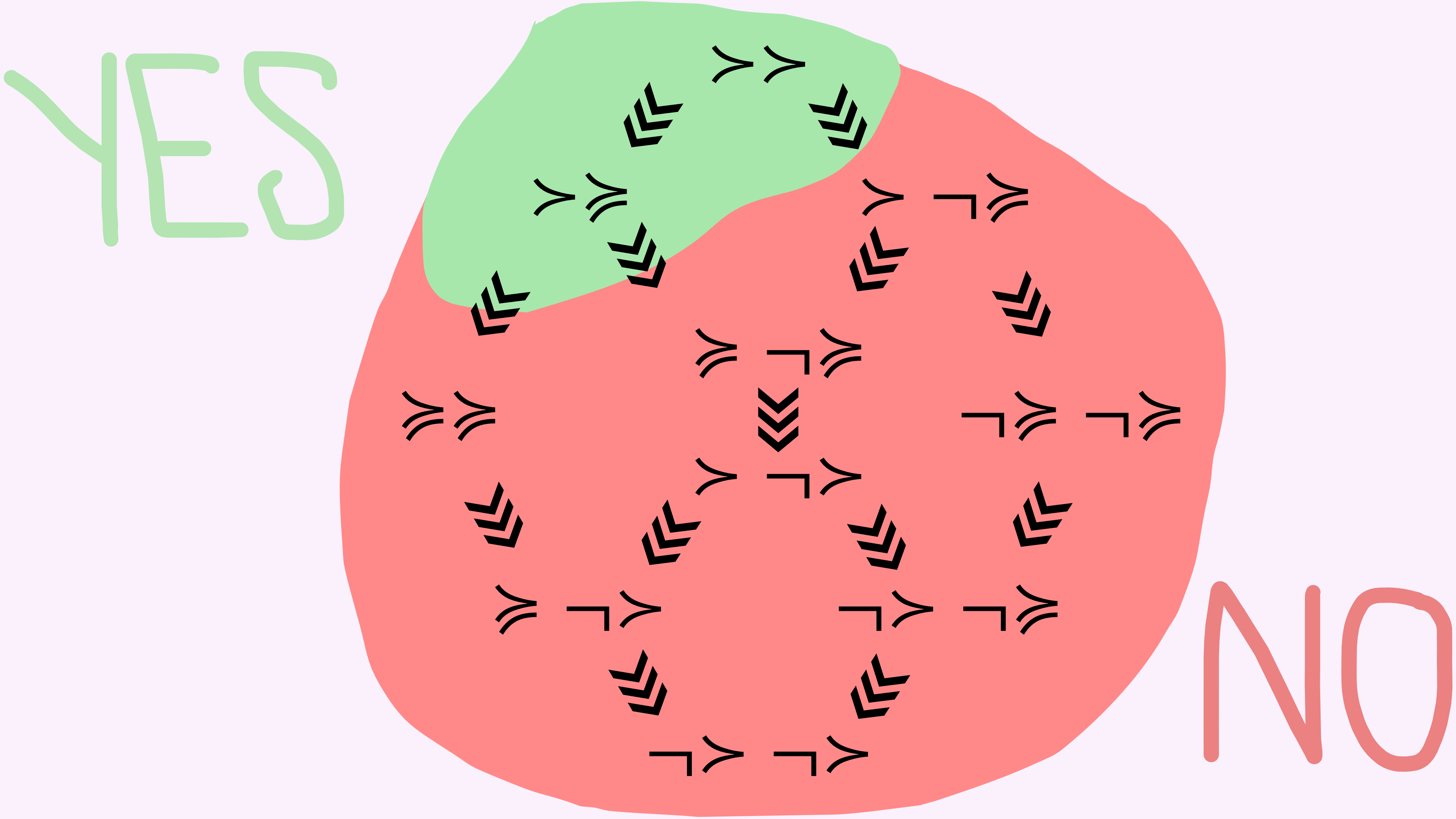
Transitivity of the majority relation

Majority relation: $A \succ_M B$ if there are more voters that prefer **A** over **B** than voters that prefer **B** over **A**.

With total orders, if preferences are single-peaked, **there is always a Condorcet winner and the majority relation is transitive** [Black, 1948]

Question: in the model with indifferences and incompleteness, for which notions of single-peakedness does this remain true?

Transitivity of the majority relation

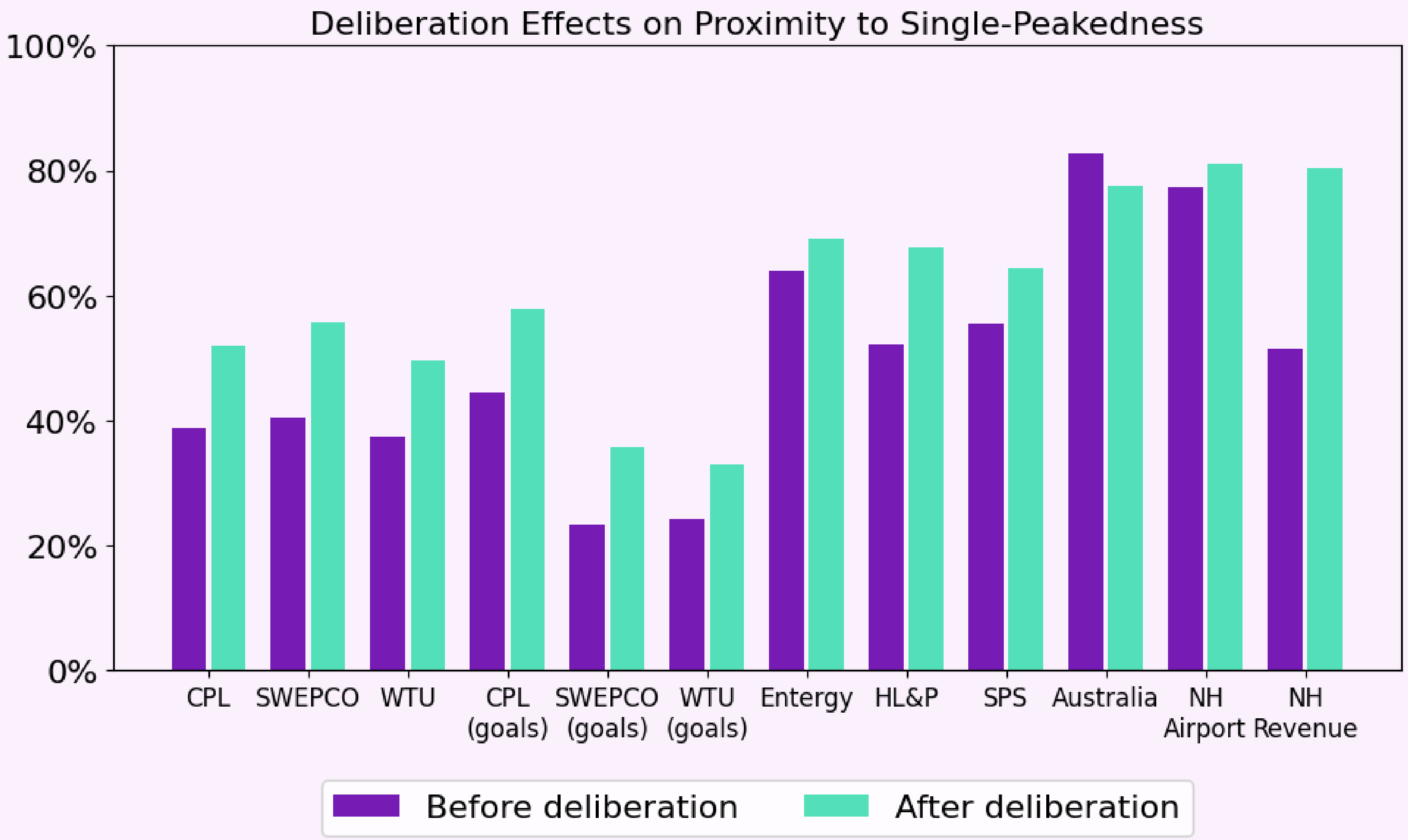


- 3 -
What Difference in Practice?

Effect of deliberative polls

$\succ\succ$

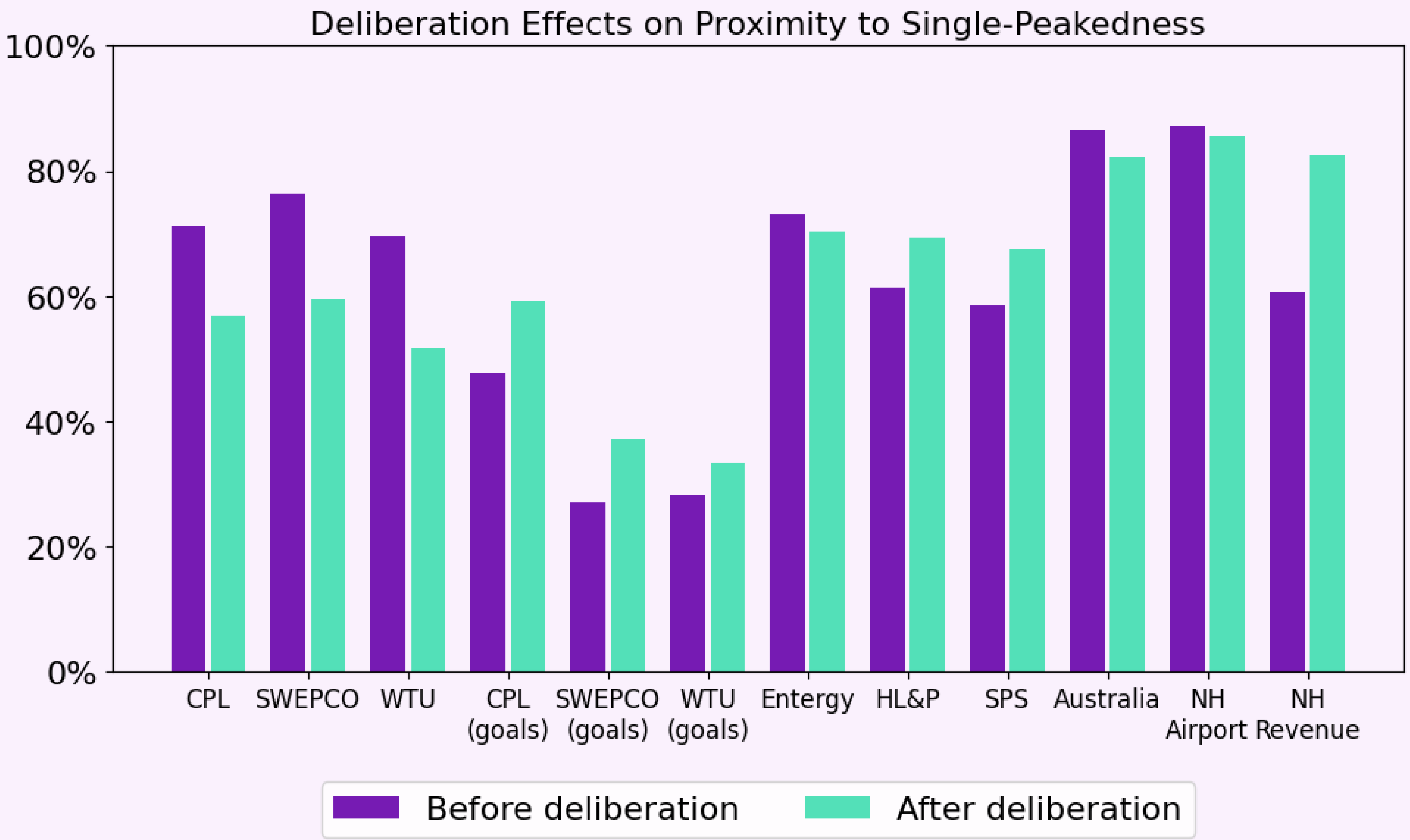
Increase in single-peakedness!



Effect of deliberative polls

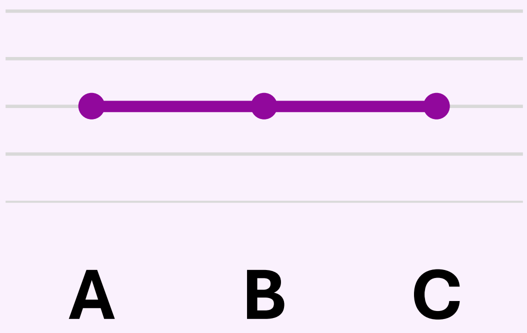
$\succ \succcurlyeq$

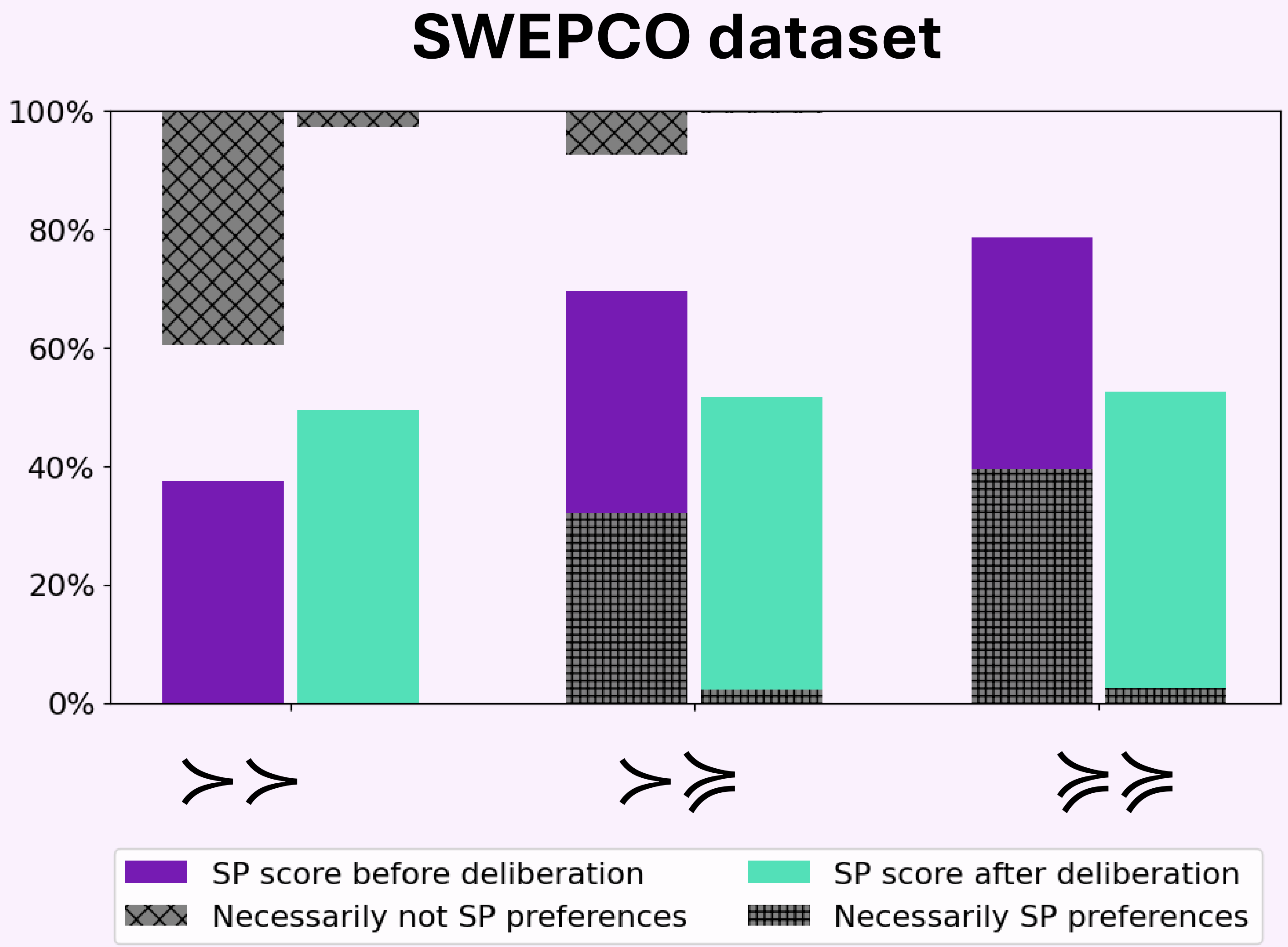
Increase in single-peakedness?



Why so much difference?

Empty preferences are **never single-peaked** for $\succ\succ$ and are **always single-peaked** for $\succ\succcurlyeq$





- 4 -
Further Work and Possibility for Collaboration

Summary and Take away

When preferences contain indifferences and/or incomparabilities, **there are several ways to define single-peakedness.**

In theory, only some of these notions **keep the nice properties** of the classical single-peakedness.

In practice, some notions are very sensitive to incomplete preferences. One **must be careful with drawing conclusions** when using only one notion.

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Appendix

Effect of deliberative polls

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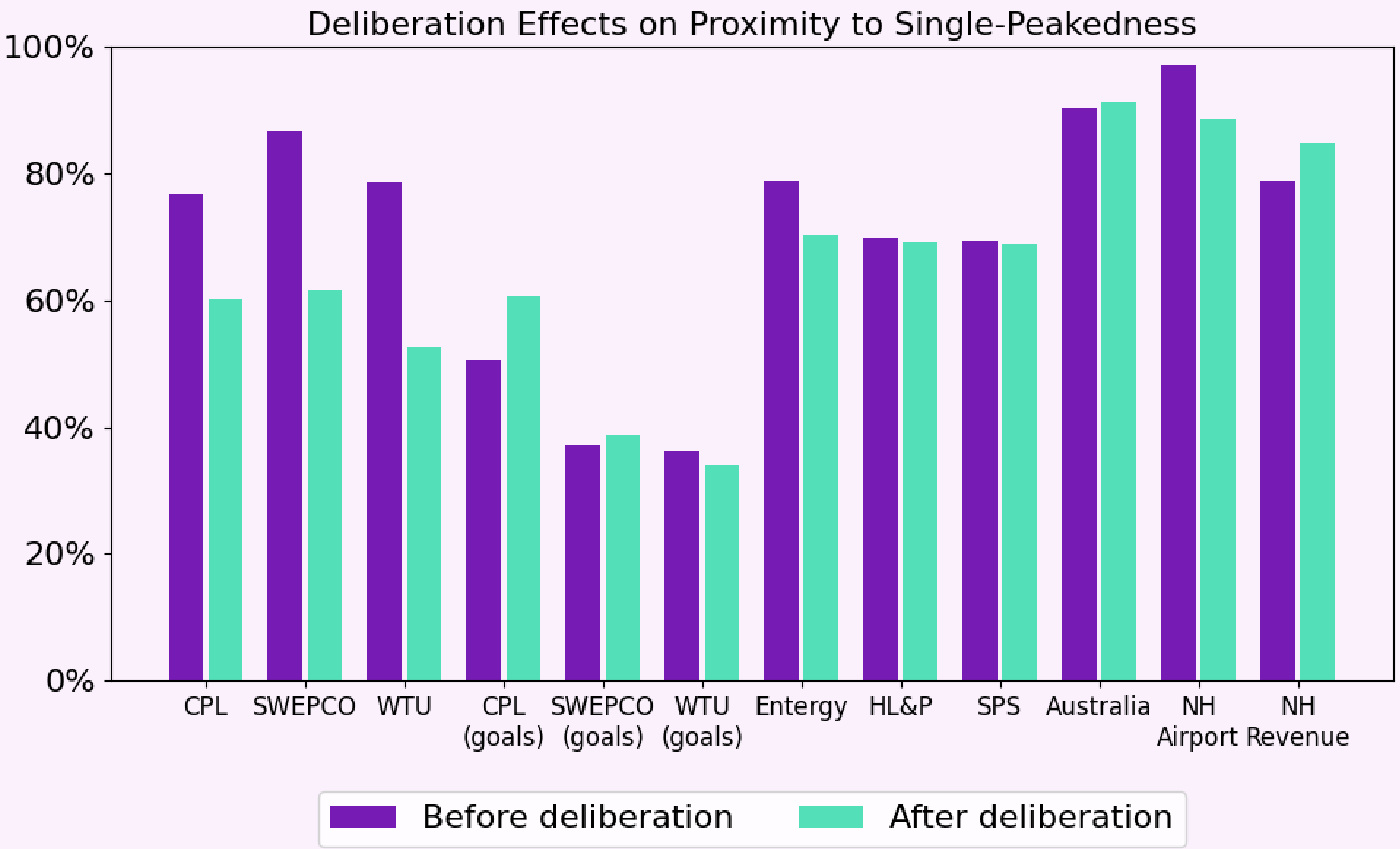
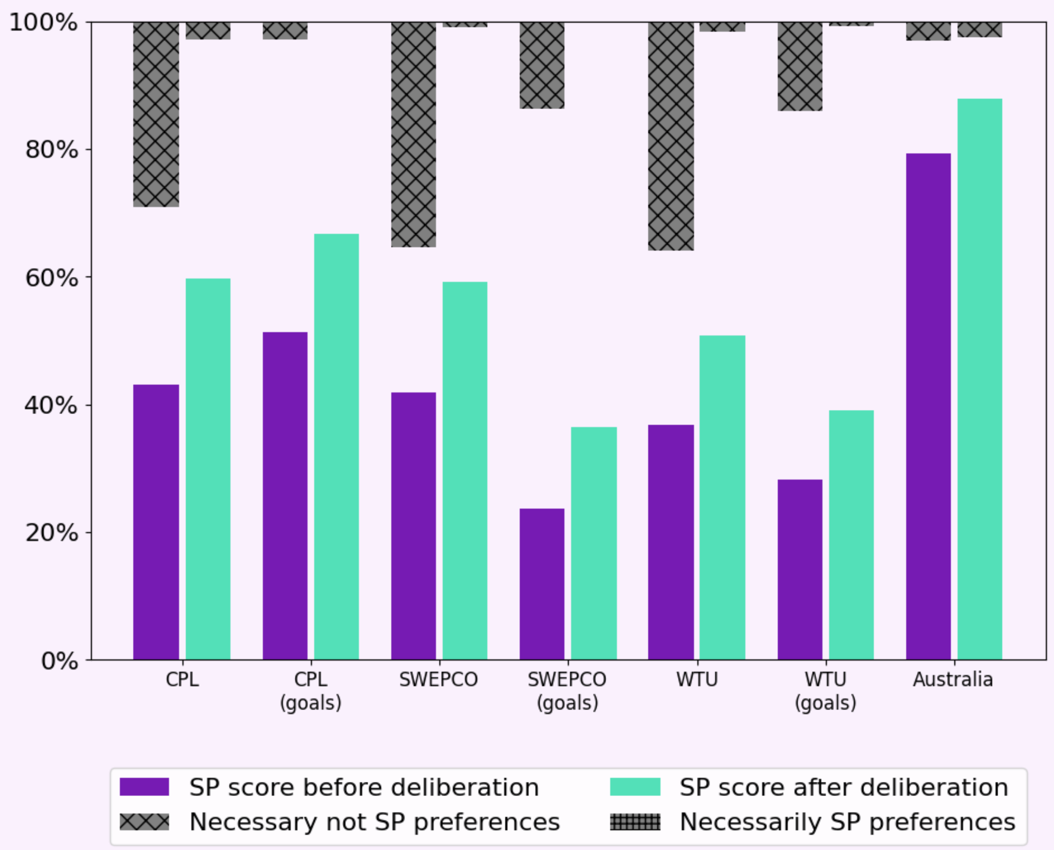
Increase in single-peakedness?

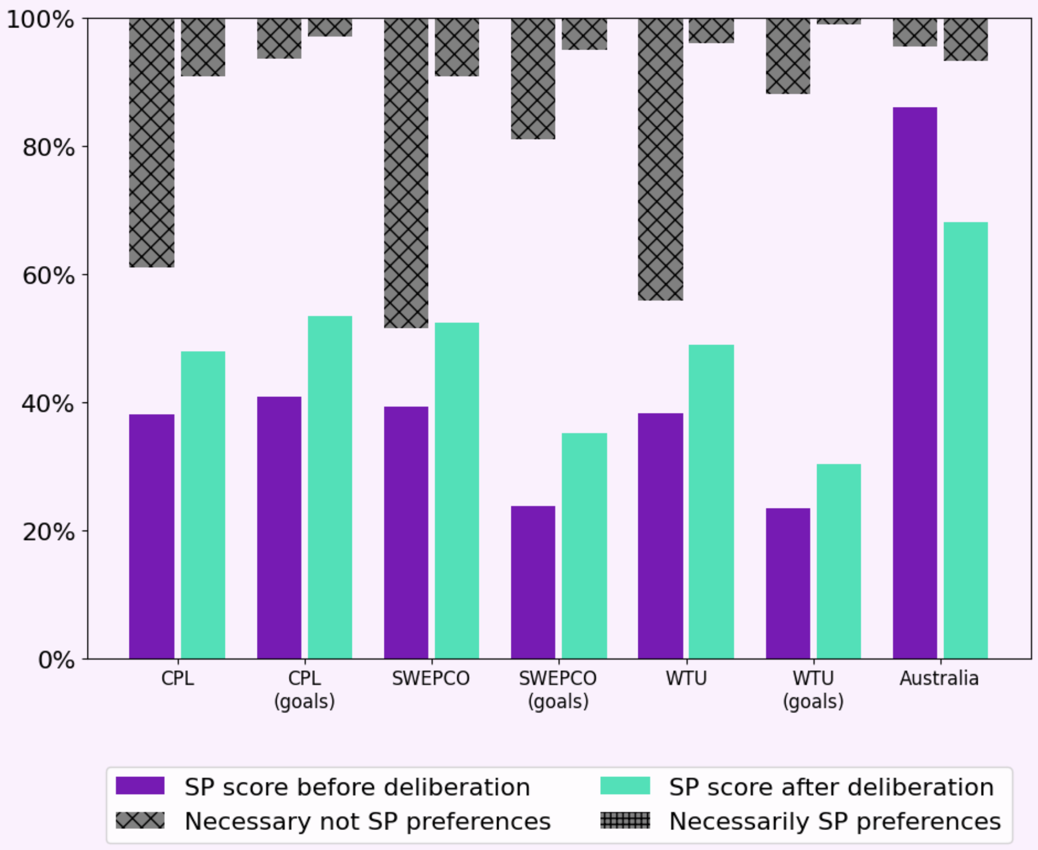


Impact of knowledge

High knowledge



Low knowledge



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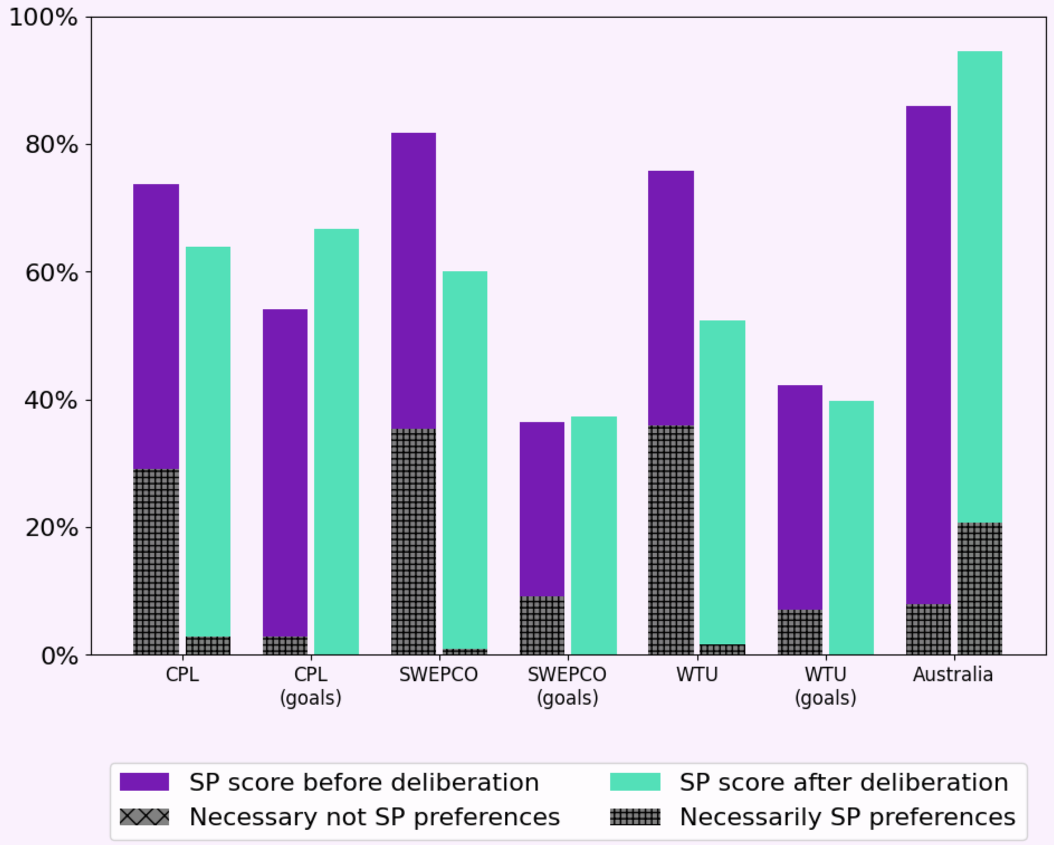

Low knowledge