Comparing Ways of Obtaining Candidate Axes from Approval Ballots

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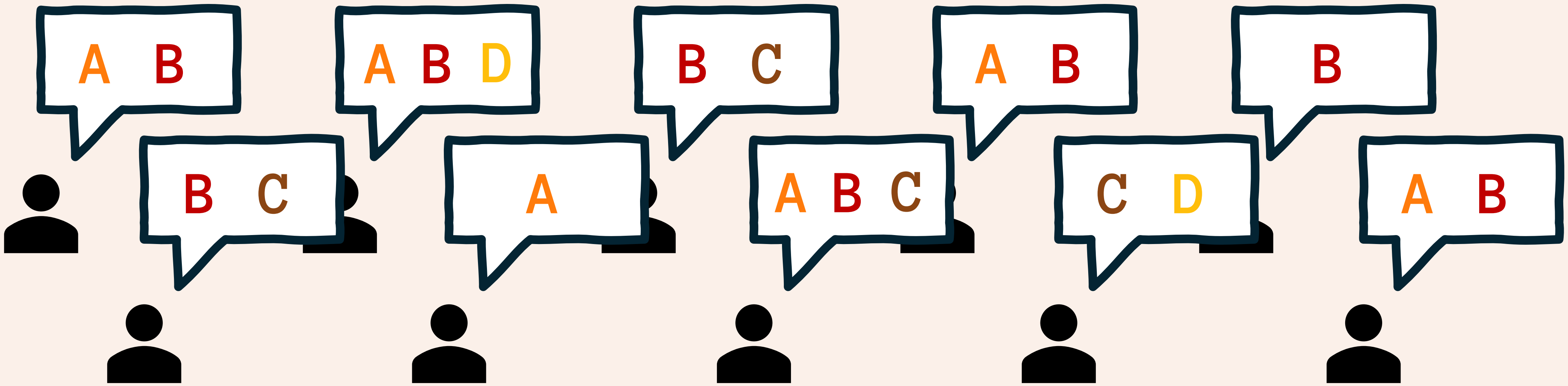
Input: Approval Ballots

Set of **candidates**: A, B, C, D

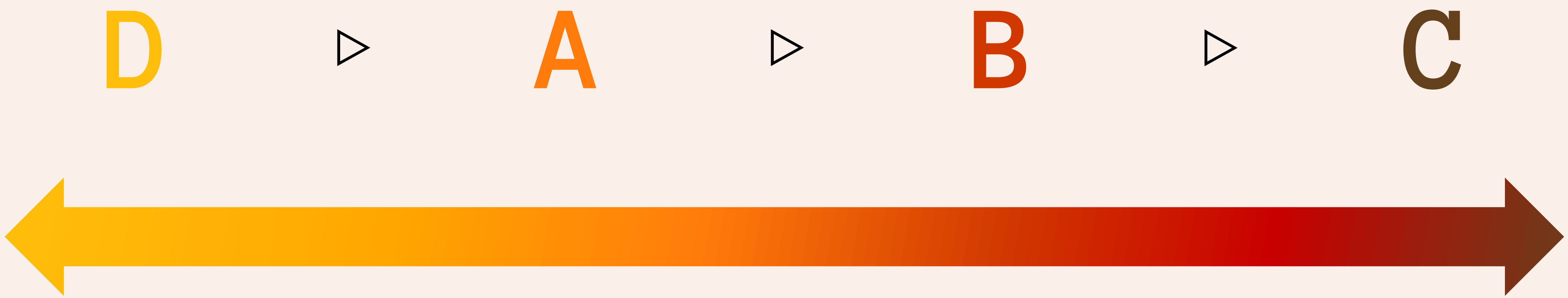
Each **voter** indicates which candidates they **approve** of:



Input: Approval Ballots

	A	B	C	D
4 ×	✓	✓		✓
4 ×	✓	✓		
3 ×	✓			✓
1 ×	✓	✓	✓	
1 ×		✓	✓	

Output: Candidate axis



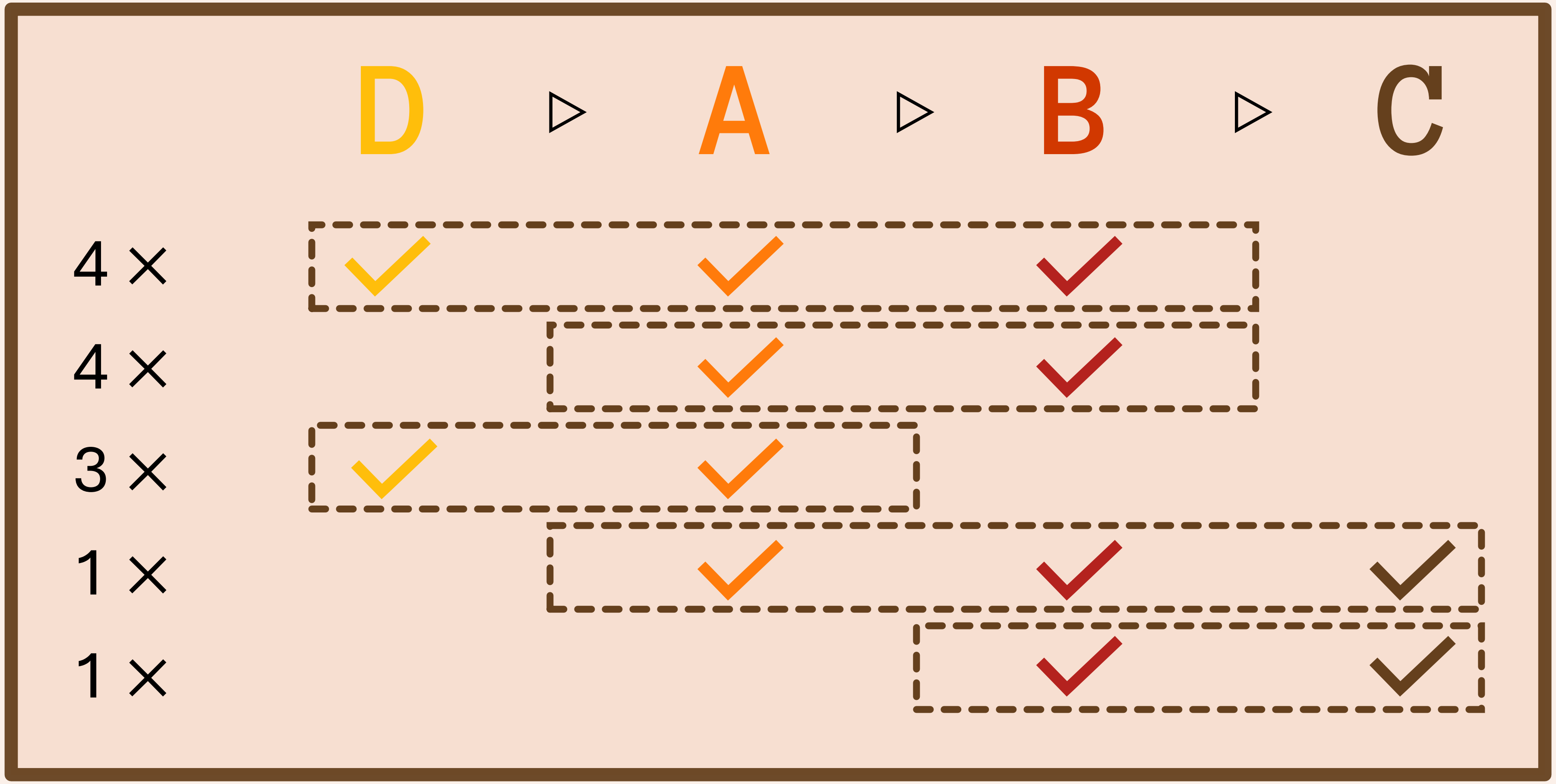
Output: Candidate axis

C ▷ B ▷ A ▷ D

The ideal case

	A	B	C	D
4 ×	✓	✓		✓
4 ×	✓	✓		
3 ×	✓			✓
1 ×	✓	✓	✓	
1 ×		✓	✓	

The ideal case

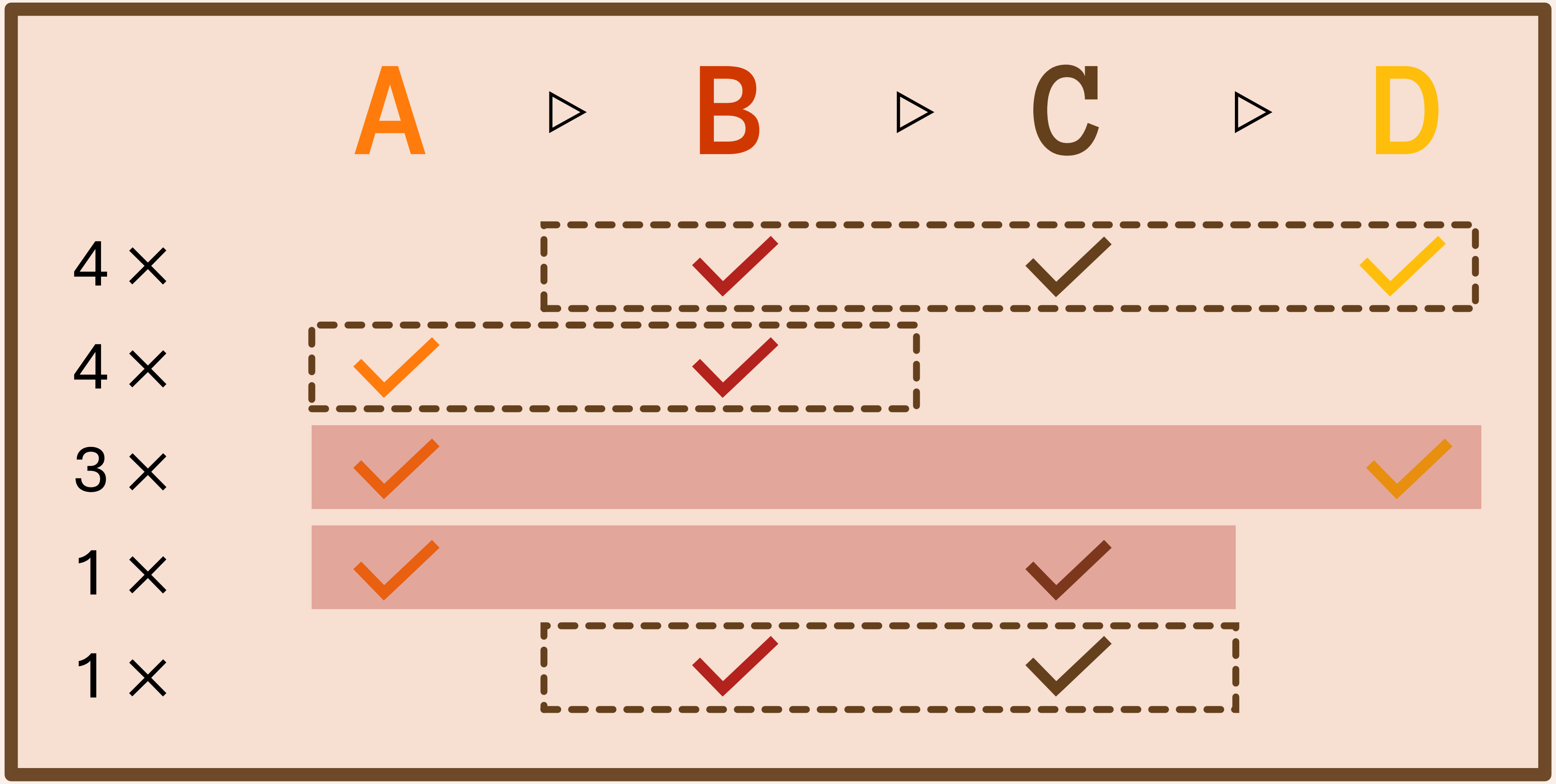


Candidate Interval (CI) property [Elkind and Lackner, 2015]

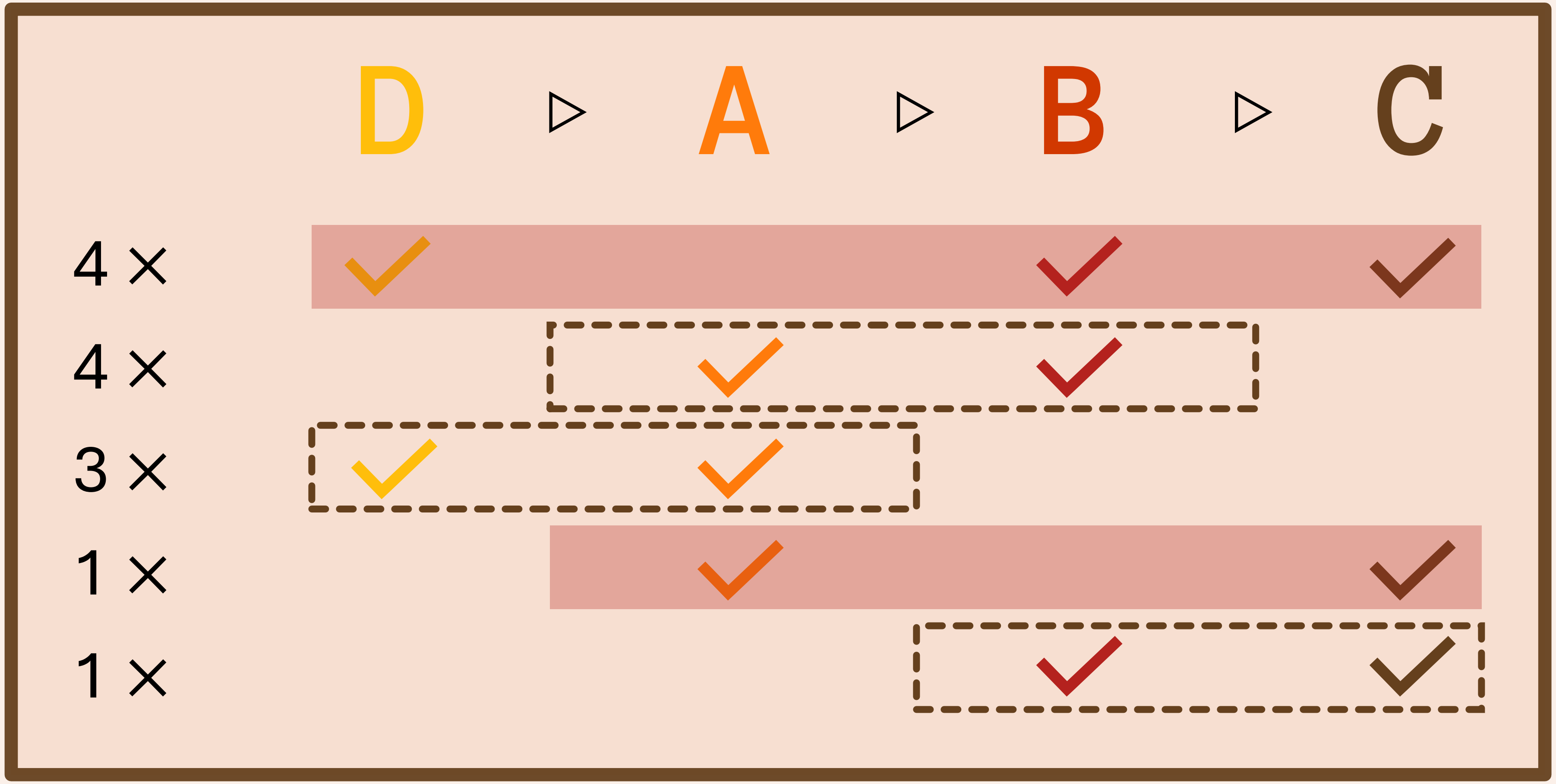
The realistic case

	A	B	C	D
4 ×		✓	✓	✓
4 ×	✓	✓		
3 ×	✓			✓
1 ×	✓		✓	
1 ×		✓	✓	

The realistic case



The realistic case



Related work: Near single-peakedness

$4 \times \quad A \succ B \succ C \succ D$

$4 \times \quad B \succ A \succ C \succ D$

$3 \times \quad C \succ D \succ B \succ A$

$1 \times \quad C \succ B \succ D \succ A$

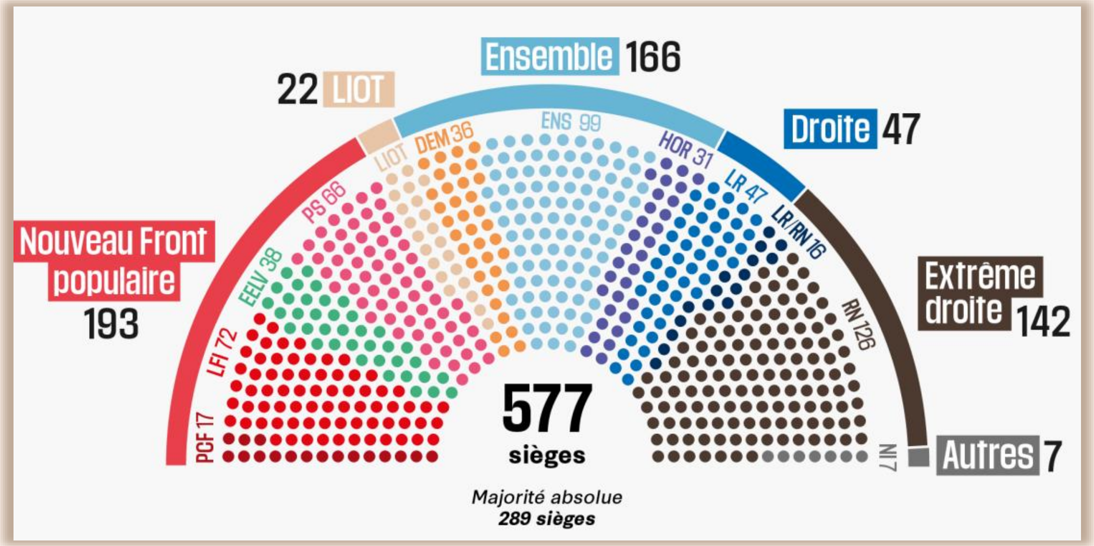
$D \rhd A \rhd B \rhd C$

Various **rules** were introduced [Niemi, 1969; Elkind and Lackner, 2014; Faliszewski et al., 2014; Erdélyi et al., 2017; Escoffier et al., 2021].

Axiomatic Analysis [Tydrichová, 2023].

Empirically, methods were not very convincing [Escoffier et al., 2021].

Motivation: Politics



How to order **political parties/politicians** based on...

...approval preferences of voters? [Lebon et al., 2017; Baujard and Lebon, 2022]

...their votes on bills in the parliament?

Motivation: Much more than politics...

Candidates

Voters

$$\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}$$

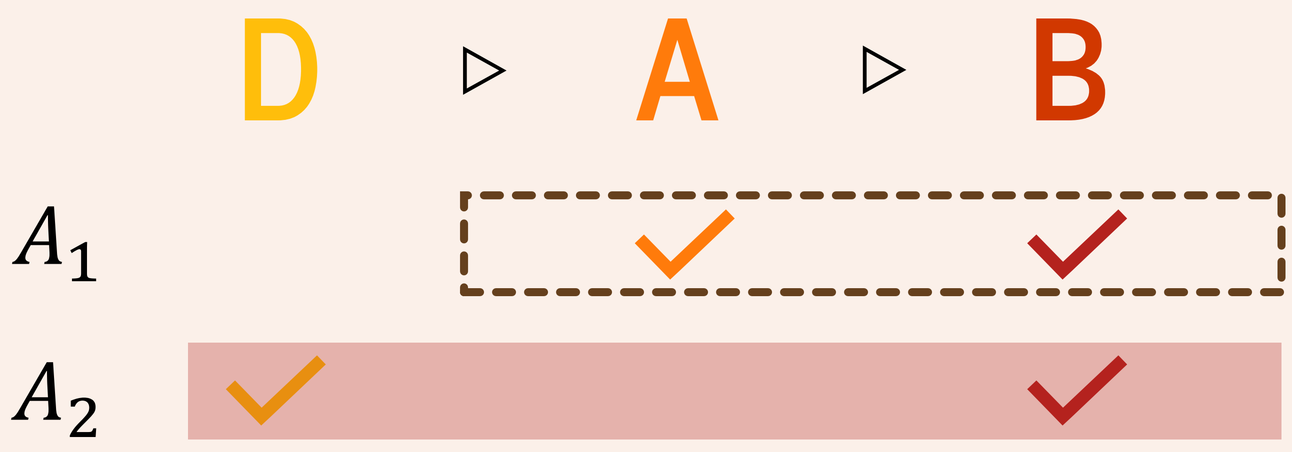
- **Parliament** (votes of **parties** on **bills**).
- **Archeological seriation** (presence of **drawing styles** on **artifacts**).
- **Geological relative dating** (presence of **fossils** in **geological strata**).
- **Scheduling** (**keywords** of **talks**).

[1] Axes rules

Formal **definitions** and notations (1)

- **n voters** $V = \{1, \dots, n\}$ and **m candidates** $C = \{c_1, \dots, c_m\}$.
- Preference profile of **Approval ballots** $P = (A_1, \dots, A_n)$ with $A_i \subseteq C$.
- We want to obtain an **Axis (ordering of candidates)** $\rhd \in L(C)$.
- The direction of an axis is irrelevant (e.g., ABCD = DCBA).
- An **axis rule** takes as input a preference profile and return a set of axes.

Formal **definitions** and notations (2)



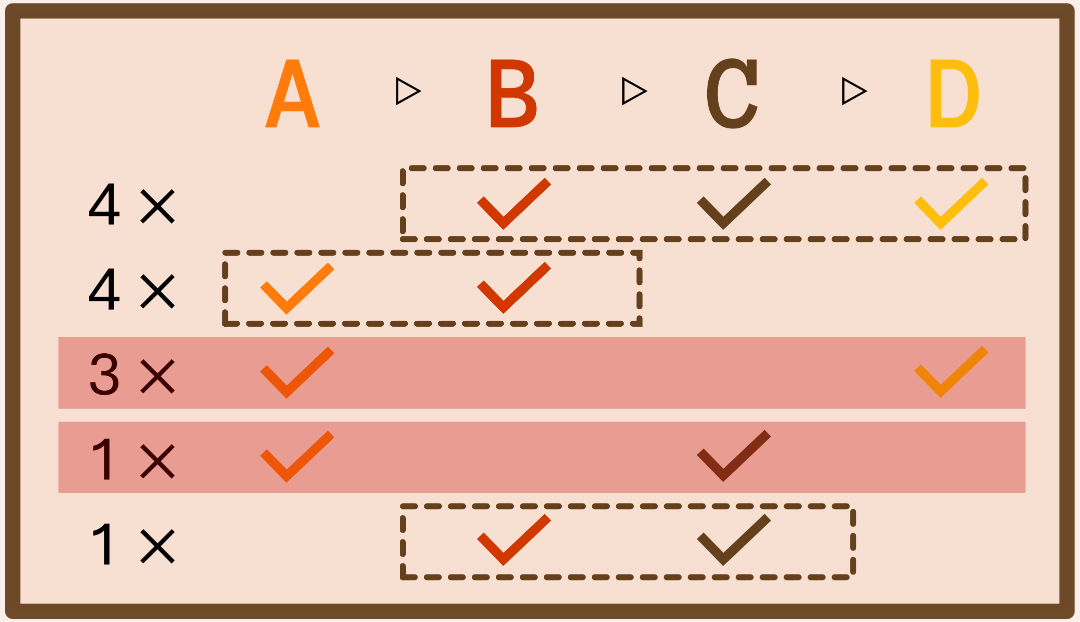
Interval of the axis

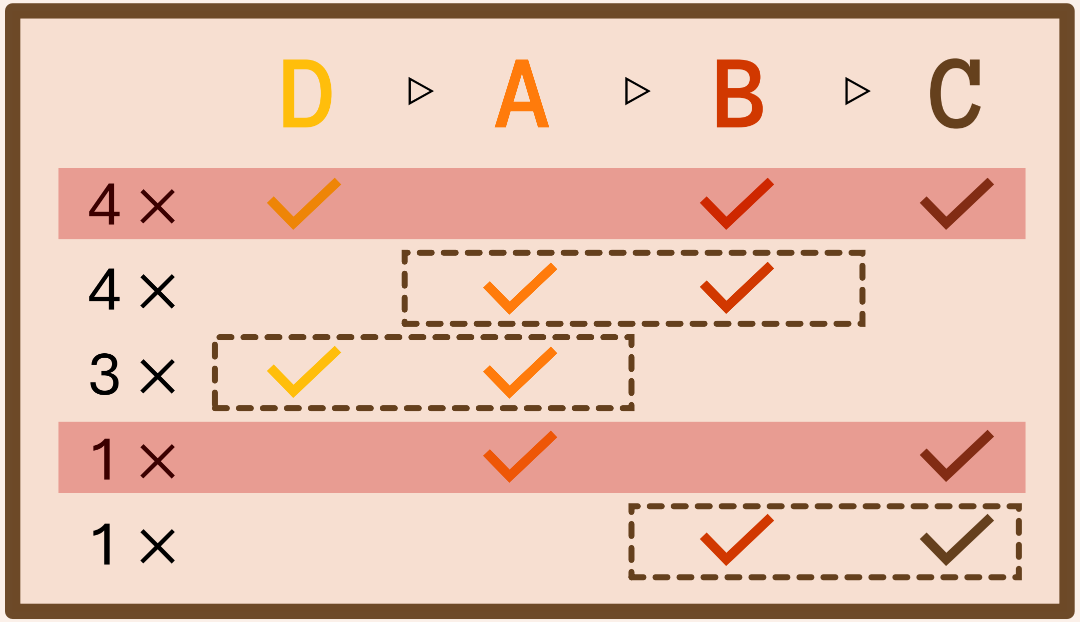
Candidate **A** is **interfering**

A ballot is an **interval** of a given axis if no candidate is **interfering** on this ballot for this axis.

Axis rules: Voter Deletion

Voter Deletion (VD): select the axes that minimize the number of ballots to delete from the profile to have only intervals of these axes.





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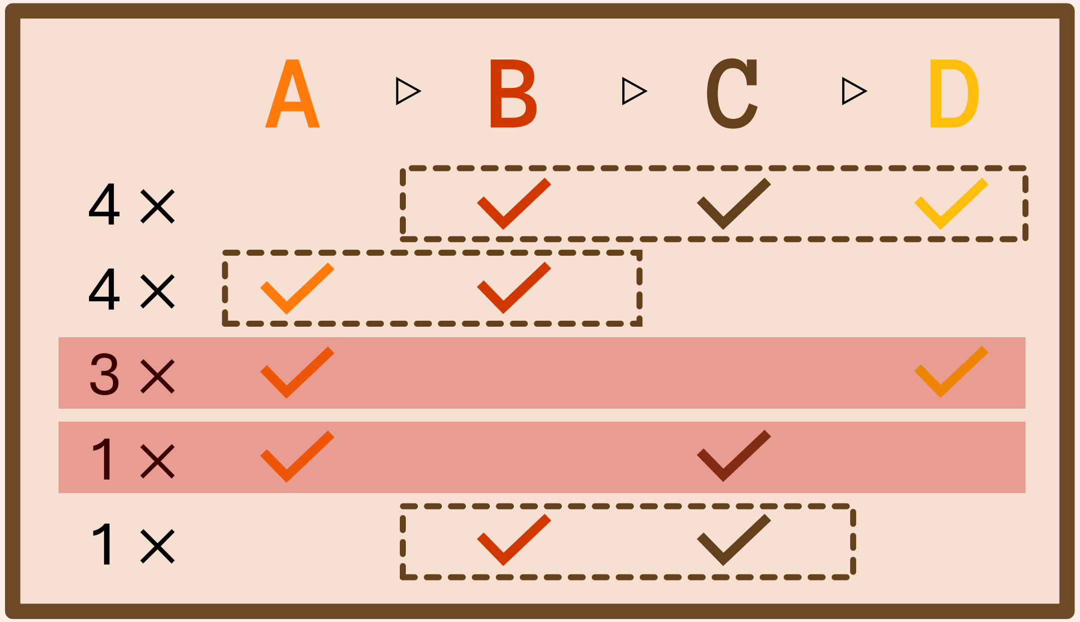
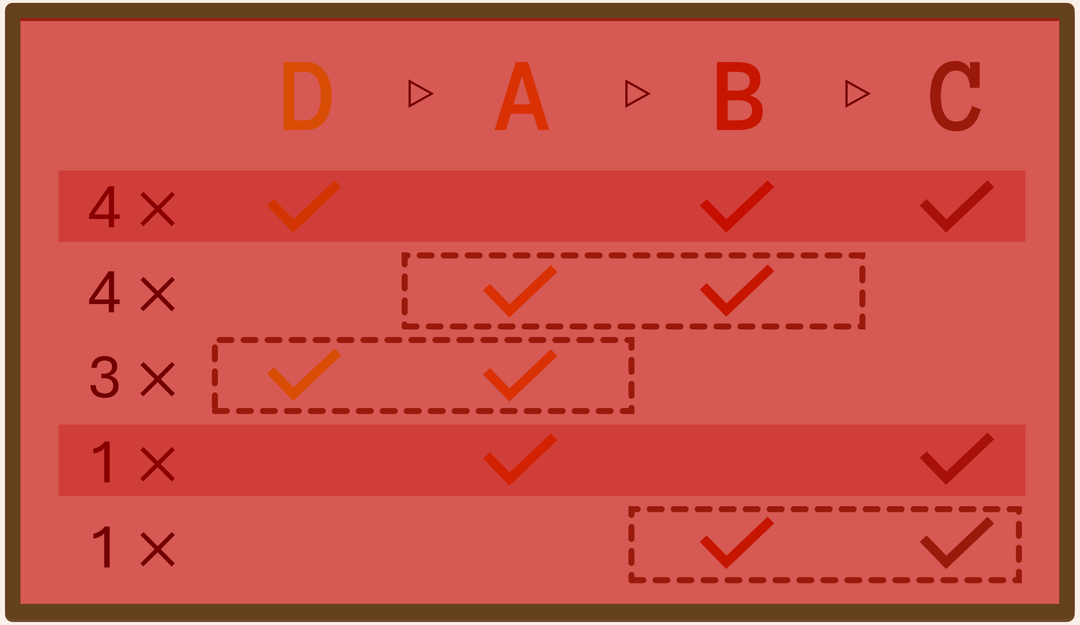
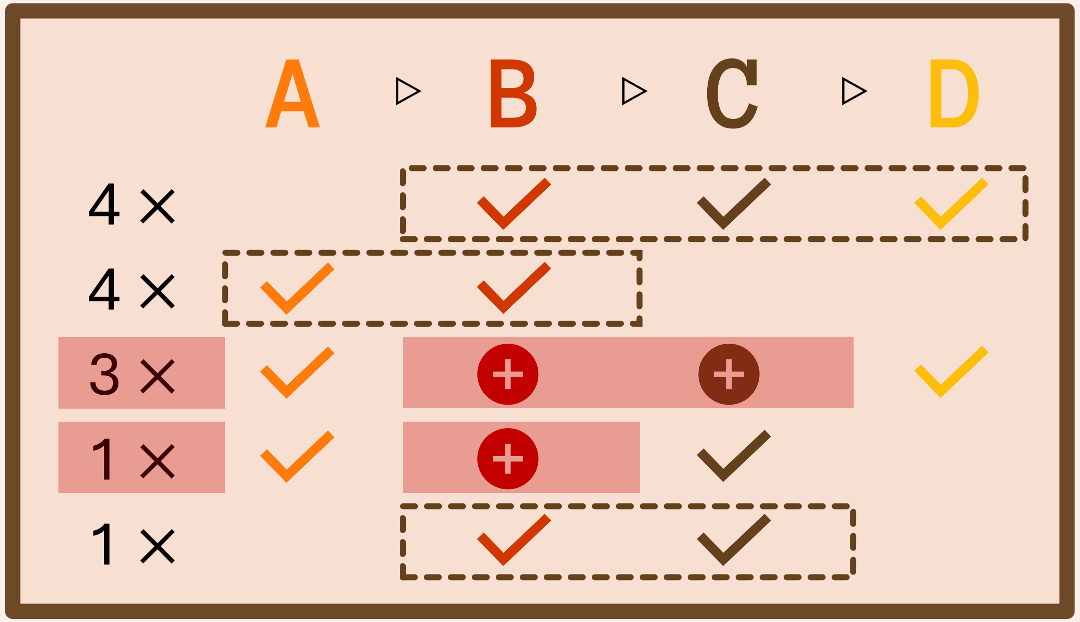
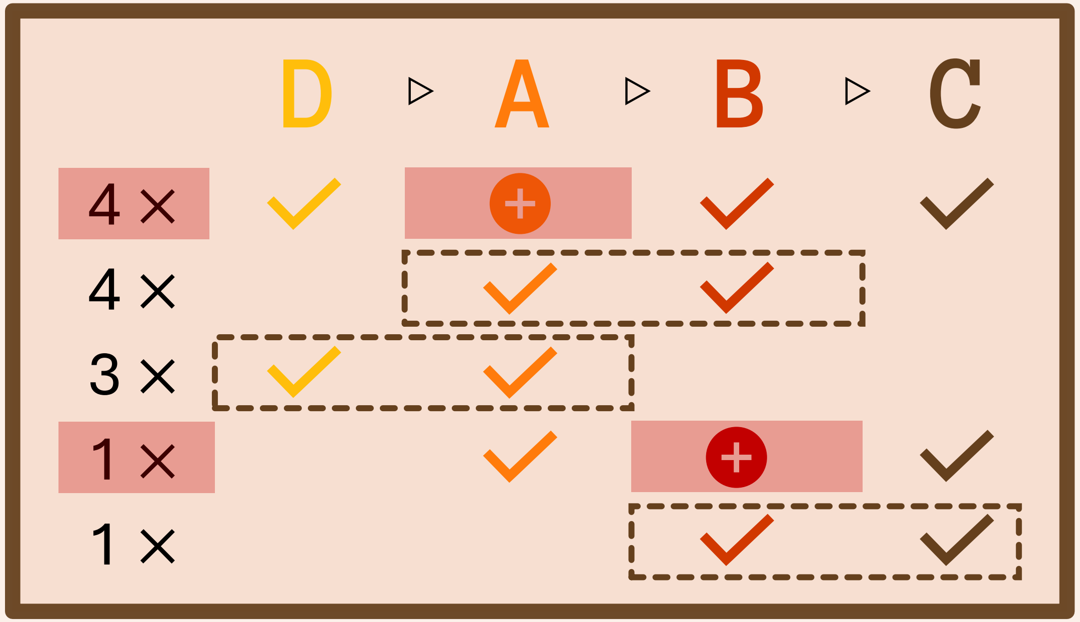




Axis rules: Ballot Completion [Lebon et al, 2017]

Ballot Completion (BC): select the axes that minimize the number of candidates to add to the approval ballots to have only intervals of these axes.





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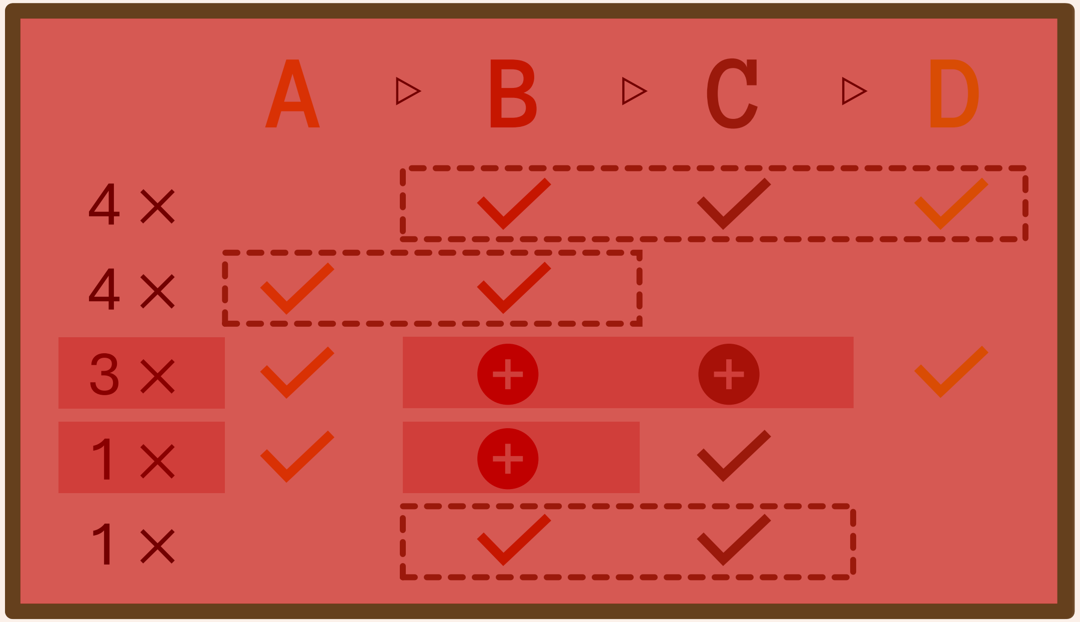
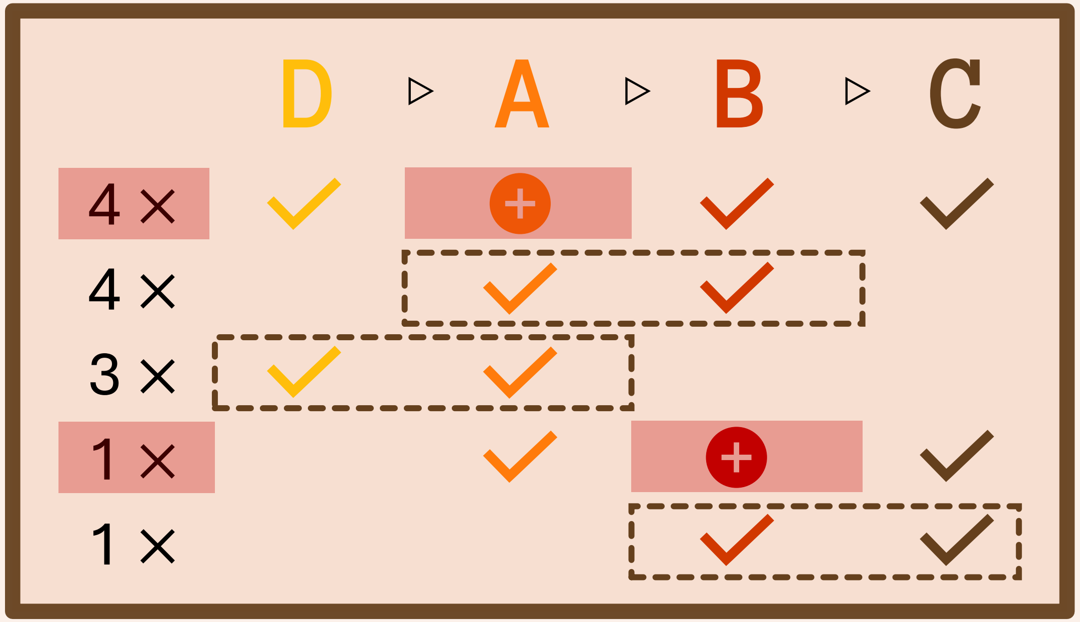




The family of Scoring rules

Scoring rules rely on a **cost function** $\mathtt{cost}(A, \rhd)$ which associates every ballot $A \subseteq C$ and axis $\rhd$ to a cost.

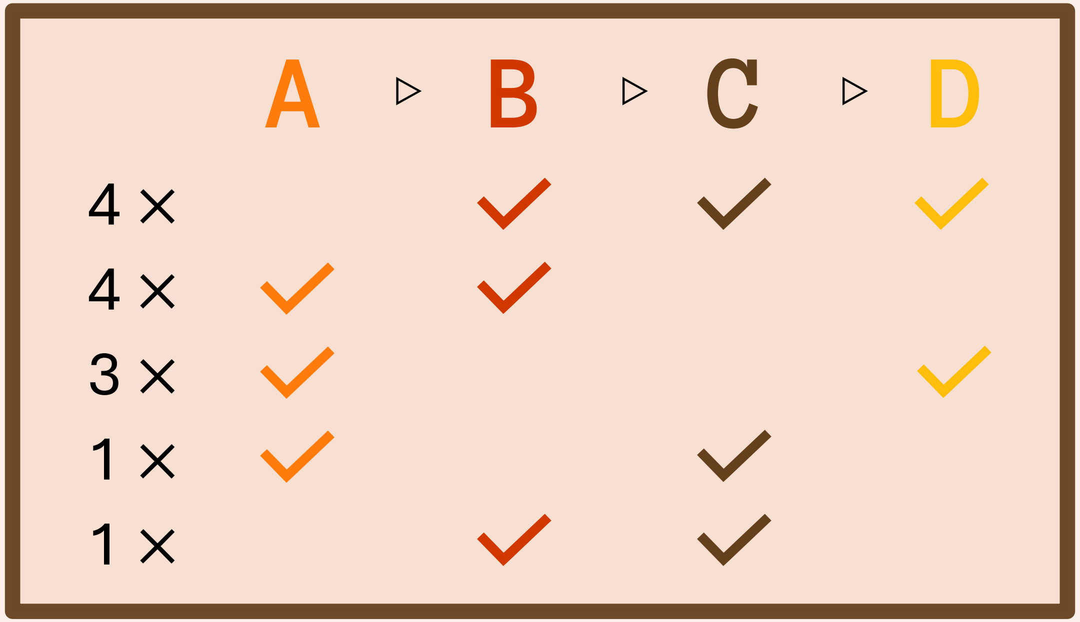
Then, they select axes **minimizing the total cost**.

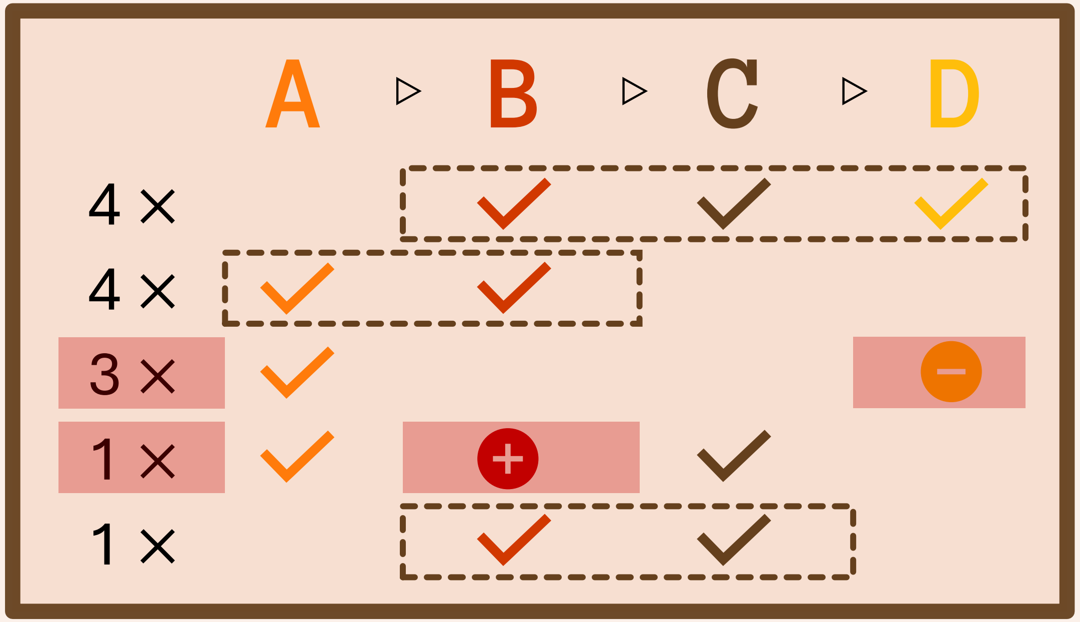
$$R(P) = argmin_{\rhd} \sum_{i \in V} \mathtt{cost}(A, \rhd).$$

- $\mathbf{cost_{VD}}(A, \rhd) = 0$ if A is an interval of $\rhd$ and 1 otherwise.
- $\mathbf{cost_{BC}}(A, \rhd) = |\{b \notin A \mid a \rhd b \rhd c$ for some $a, c \in A\}|$.

Axis rules: Minimum Flips

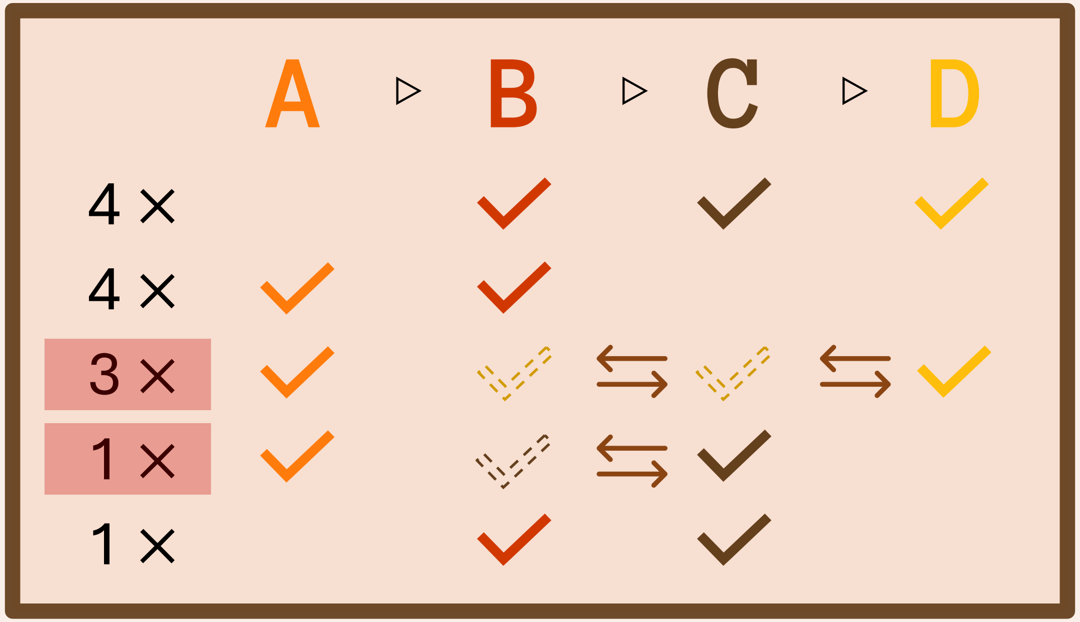
Minimum Flips (MF): select the axes that minimize the number of candidates to add **or remove** to the approval ballots to have only intervals of these axes.





Axis rules: Minimum Swaps

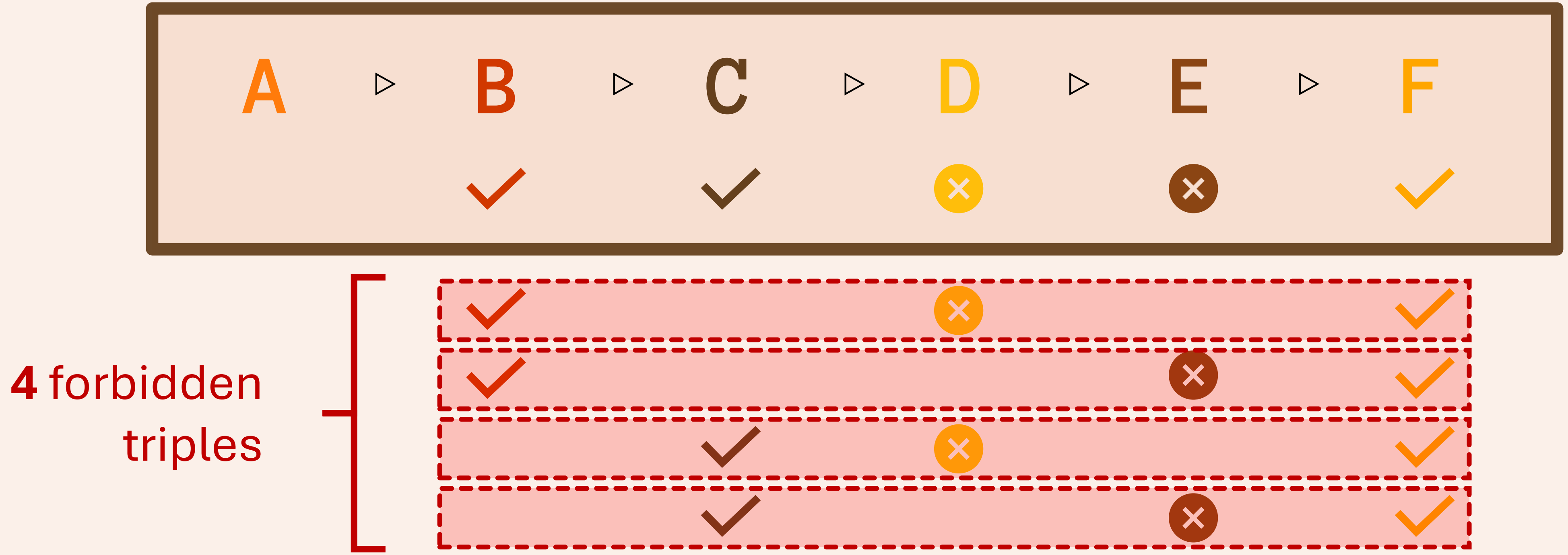
Minimum Swaps (MS): the cost of a ballot for an axis is the number of swaps of adjacent candidates on the axis that must be preformed for the approval ballot to be an interval of the axis.



Total cost: 7 swaps

Axis rules: Forbidden Triples

Forbidden Triples (FT): select the axes that minimize the total number of Forbidden Triples.



Relationship between cost functions

For 3 candidates…

VD = MF = BC = MS = FT

For 4 candidates…

VD = MF ≤ BC = MS ≤ FT

For $m \geq 5$ candidates…

VD ≤ MF ≤ BC ≤ MS ≤ FT

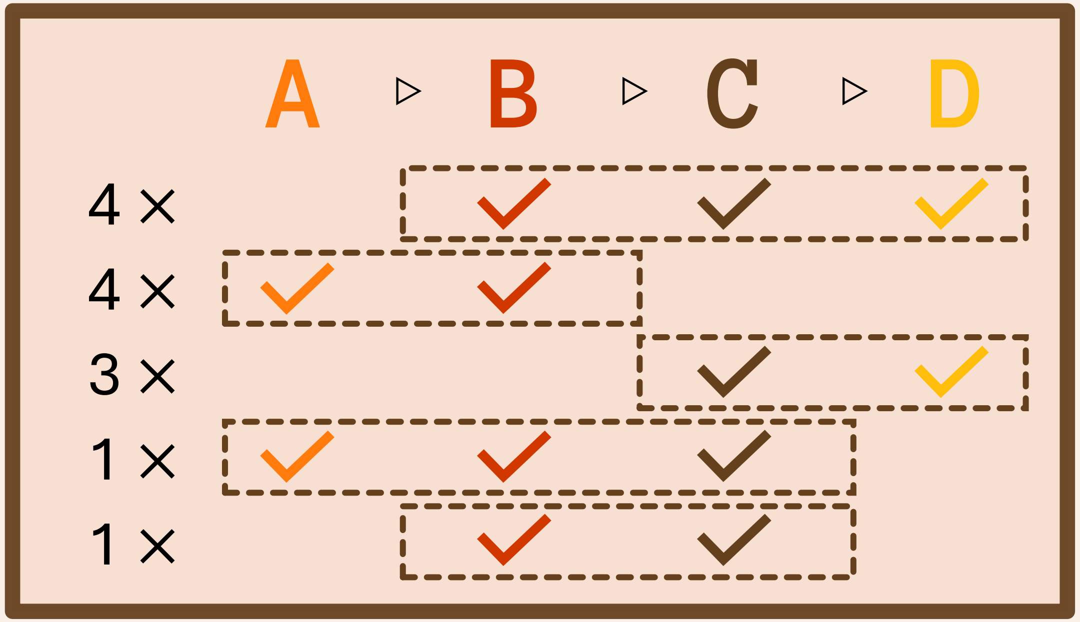
Complexity of axis rules

It is **NP-hard** to compute the optimal axes for any of these rules, even in profiles in which every voter approves at most 2 candidates [Booth, 1975].

[2] Axiomatic Analysis

Axiom: Consistency with linearity

An axis rule satisfies **consistency with linearity** if whenever there exists an axis for which all ballots are interval, it return **all** such axes.

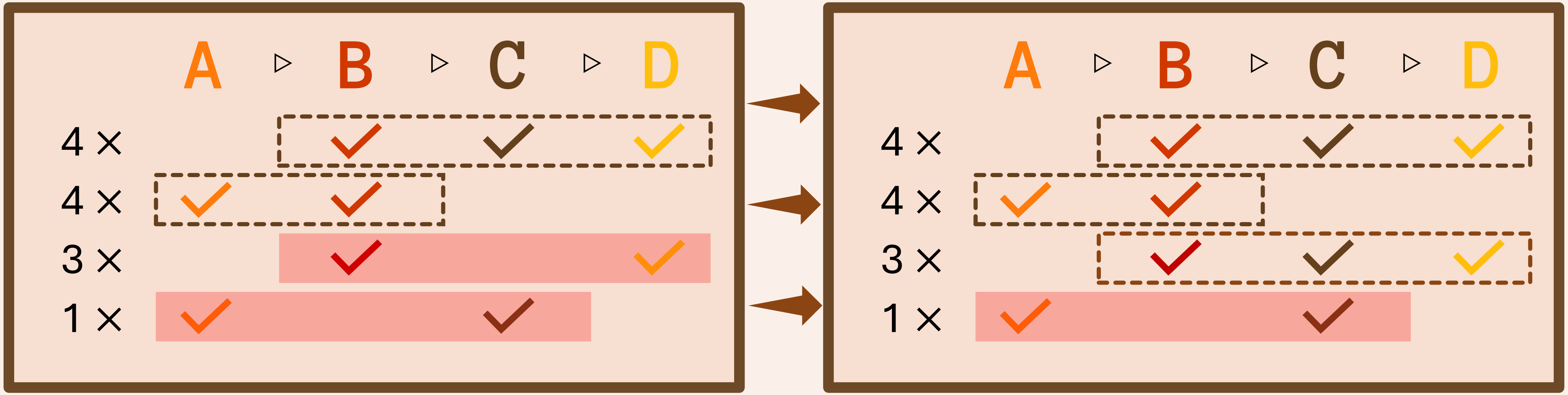


VD
MF
BC
MS
FT

Axiom: Monotonicity

An axis rule satisfies **ballot monotonicity** if whenever an axis $\rhd$ is selected for a profile P, it is still selected if we replace a non-interval ballot A by:

$$A' = A \cup \{x \in C \mid \exists a, b \in A, a \rhd x \rhd b\}$$



Axiom: Monotonicity

Voter Deletion (VD) and **Ballot Completion** (BC) satisfy this axiom, but not MS, MF and FT.

Proof idea (Voter Deletion).

- For all axes, we are only changing the cost of the altered ballot.
- For Voter Deletion, this cost can change of at most 1.
- The cost of the optimal axis goes from 1 to 0 (since the ballot is now an interval).
- Thus, the cost of optimal axis remains minimal.

Axiom: Monotonicity

Voter Deletion (VD) and **Ballot Completion** (BC) satisfy this axiom, but not MS, MF and FT.

Proof idea (Ballot Completion).

- For all axis, we are only changing the cost of the altered ballot.
- For Ballot Completion, this cost can change of **at most k** (the number of interfering candidates on the altered ballot).
- The cost of the optimal axis goes **from k** to 0 (since the ballot is now an interval).
- Thus, the cost of optimal axis remains minimal.

Definition: clones

	A	B	C	D	β
4 ×	✓	✓		✓	✓
4 ×	✓	✓			✓
3 ×	✓			✓	
1 ×	✓	✓	✓		✓
1 ×		✓	✓		✓

Axiom: Independence of clones

An axis rule satisfies **independence of clones** if an axis $\rhd$ is selected for a profile P containing clones if and only if its reduction $\rhd_{-\beta}$ when removing one of the clones is selected in the profile $P_{-\beta}$ in which this clone is removed.

	A	B	β	C	D
4 ×		✓	✓	✓	✓
4 ×	✓	✓	✓		
3 ×		✓	✓		✓
1 ×	✓			✓	

A $\rhd$ B $\rhd$ β $\rhd$ C $\rhd$ D

is selected.

Axiom: Independence of clones

An axis rule satisfies **independence of clones** if an axis $\triangleright$ is selected for a profile P containing clones if and only if its reduction $\triangleright_{-\beta}$ when removing one of the clones is selected in the profile $P_{-\beta}$ in which this clone is removed.

	A	B	C	D
4 ×		✓	✓	✓
4 ×	✓	✓		
3 ×		✓		✓
1 ×	✓		✓	

A $\triangleright$ B $\triangleright$ C $\triangleright$ D

is selected.

Characterization of Voter Deletion

Voter Deletion (VD) is the **only neutral axis scoring rule** that satisfies (1) consistency with linearity, (2) ballot monotonicity and (3) independence of clones.

Axiom: Clone-proximity

An axis rule satisfies **clone-proximity** if for any profile P in which two candidates a and a' are clones, then for all optimal axis $\rhd$, if there exists a candidate c such that $a \rhd c \rhd a'$, then c is always approved when a and a' are approved.

$A \rhd \boxed{B \rhd \beta} \rhd C \rhd D$ → Okay

$A \rhd \boxed{B} \rhd C \rhd D \rhd \boxed{\beta}$ → Not okay

Incompatibility of clone axioms

There is no neutral axis scoring rule that satisfies (1) consistency with linearity, (2) independence of clones and (3) clone-proximity.

- **Voter Deletion** satisfies independence of clones, but not the other rules.
- **Forbidden Triples** satisfies clone-proximity, but not the other rules.
- Failure to satisfy these axioms is **mainly due to ties**: rules always return the *"correct"* axis, but sometimes additionally return *"incorrect"* ones.

Centrists and Outliers

The axis rules we defined tend to push **popular** candidates towards the center, and **unpopular** ones towards the extremes.

Is it a ***bug*** or a ***feature***?

Is this phenomenon **more important for some rules**?

Centrists and Outliers

An axis rule satisfies **clearance** if for any selected axis $\triangleright$, a never-approved candidates is never interfering with any ballots (a natural position is thus at the extremes).

❌ **VD** ❌ **MF** ✅ **BC** ✅ **MS** ✅ **FT**

An axis rule satisfies **veto-winner centrism** if for any profile in which every voter approve **all but one candidate**, for any selected axis $\triangleright$, the centrist candidate is the most approved one.

❌ **VD** ❌ **MF** ❌ **BC** ✅ **MS** ✅ **FT**

Summary of axiomatic analysis

	VD	MF	BC	MS	FT
Consistency with Linearity	✓	✓	✓	✓	✓
Ballot Monotonicity	✓		✓		
Independence of Clones	✓				
Clone-proximity	*				✓
Clearance			✓	✓	✓
Veto-winner centrism				✓	✓

Summary of axiomatic analysis

	MF	VD	BC	MS	FT
Consistency with Linearity	✓	✓	✓	✓	✓
Ballot Monotonicity		✓	✓		
Independence of Clones		✓			
Clone-proximity					✓
Clearance			✓	✓	✓
Veto-winner centrism				✓	✓

Summary of axiomatic analysis

	MF	VD	BC	MS	FT
Independence of clones		✓			
Ballot Monotonicity		✓	✓		
Consistency with linearity	✓	✓	✓	✓	✓
Clearance			✓	✓	✓
Veto-winner centrism				✓	✓
Clone-proximity					✓

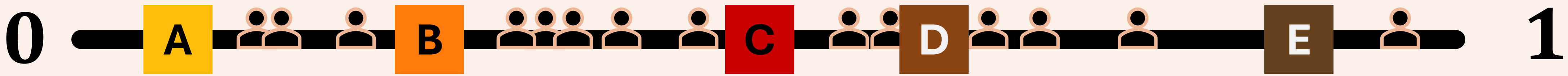
[3] Experiments

Outline of experiments

- 1D Euclidean model
- French presidential elections
- Supreme Court of the US
- Applause in French Parliament
- Tier-lists: pop-culture and colors

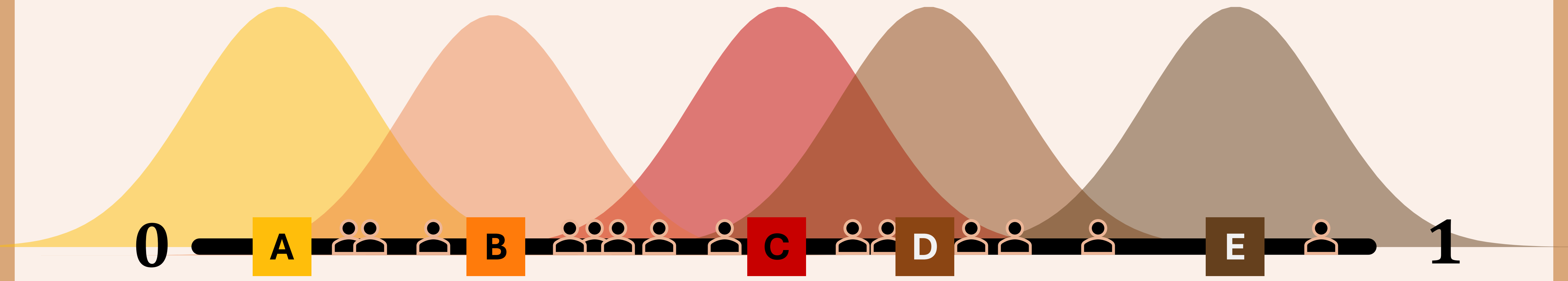
Experiments on Euclidean data

- **Voters** and **candidates** A have positions on the line (selected uniformly at random).
- Positions of candidates define a **natural axis.**
- Voters prefer (and approve) candidates that are **closer to them.**



Experiments on Euclidean data

- Voters have **noisy observations** of the candidates' positions with normal noise of variance $\sigma \in \{0.1, 0.2, 0.3, 0.4\}$.



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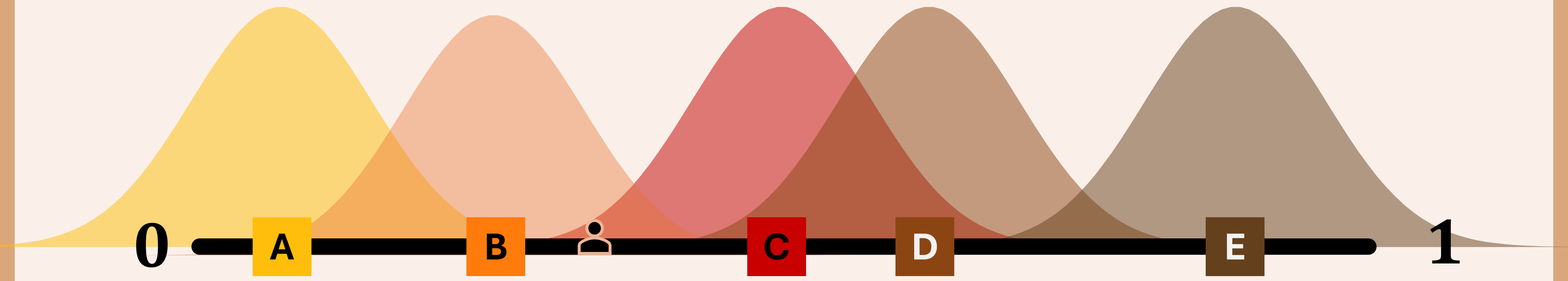
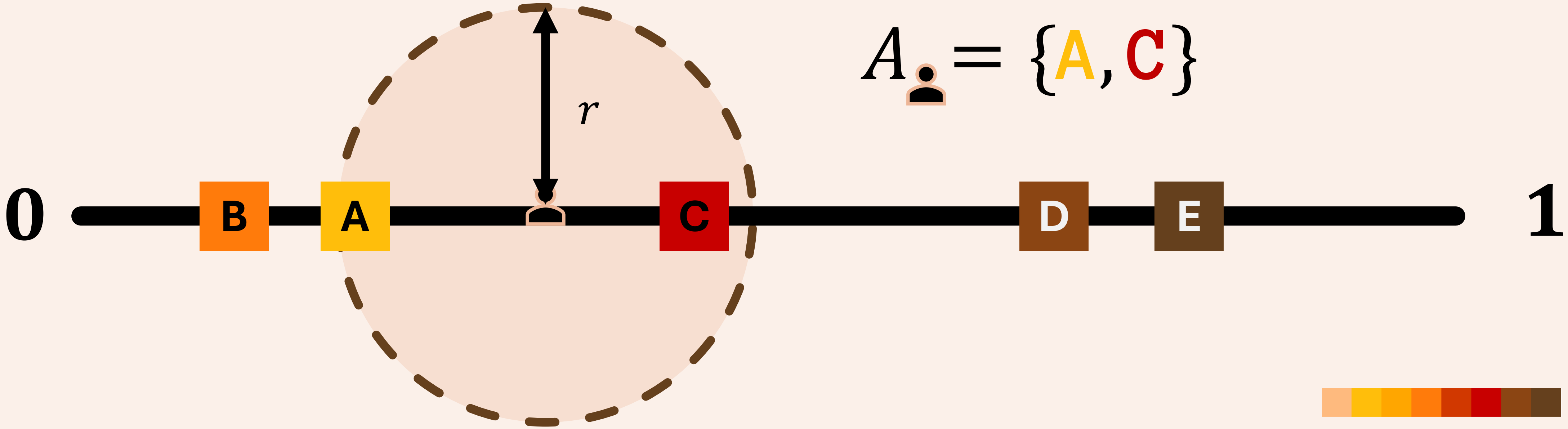


Experiments on Euclidean data

- Voters have **noisy observations** of the candidates' positions with normal noise of variance $\sigma \in \{0.1, 0.2, 0.3, 0.4\}$.
- Voters **approve** candidates that are at distance $0.2 \leq r \leq 0.6$ of them.



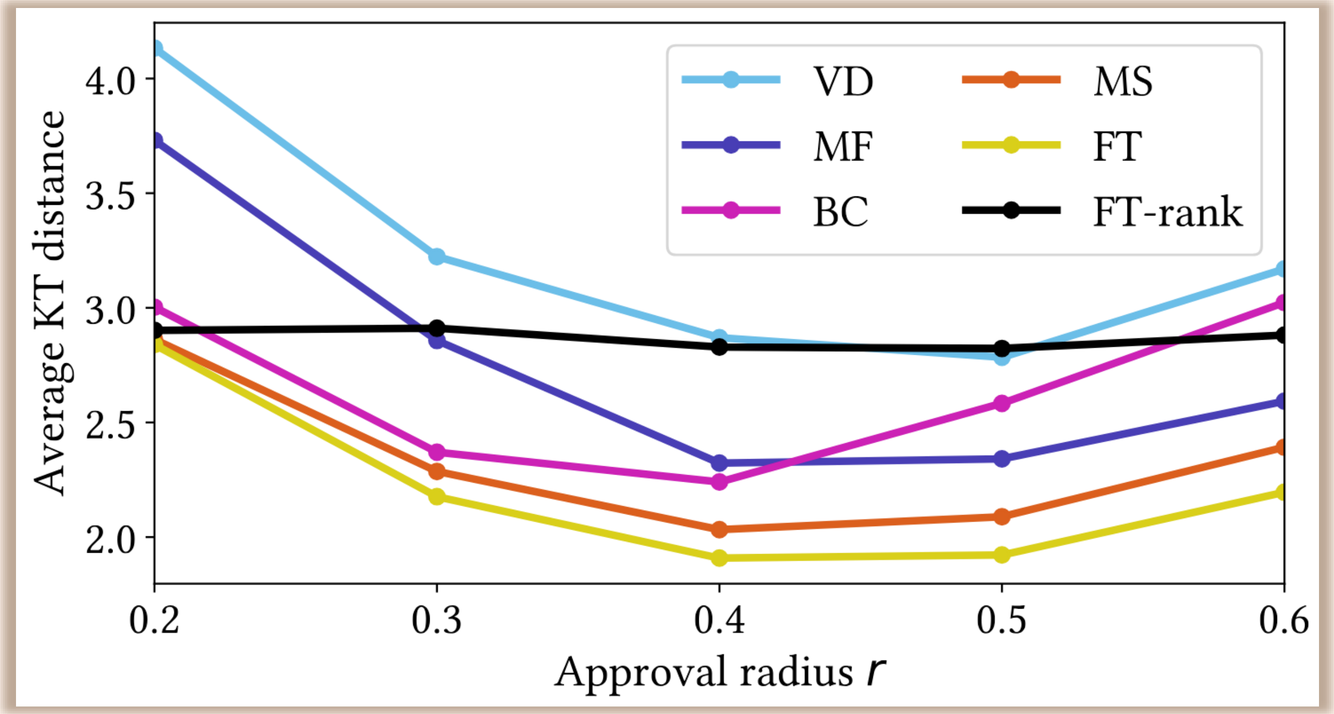
Experiments on Euclidean data

- Voters have **noisy observations** of the candidates' positions with normal noise of variance $\sigma \in \{0.1, 0.2, 0.3, 0.4\}$.
- Voters **approve** candidates that are at distance $0.2 \leq r \leq 0.6$ of them.
- Voters **rank** candidates in decreasing distance order.

C ≻ A ≻ B ≻ D ≻ E

Experiments on Euclidean data

- We measure the **average Kendall-tau distance** between the axes returned by the rules and the **ground truth axis** (here $\sigma = 0.3$).



$n = 100$ voters.

$m = 7$ candidates.

1000 iterations.

French presidential elections

Goal: ordering candidates to the French presidential elections.

Data: approval preferences collected in voting experiments between 2007 and 2022.

These datasets can be found at **theo.delemazure.fr/datasets.**

French presidential elections

- Axes returned for the dataset of the **2017 Online experiment**.

Voter Deletion	R	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	SP
Minimal Flips	LO	NPA	LFI	PS	EM	LR	DLF	RN	R	UPR	SP
Ballot Completion	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	R	SP
Minimal Swaps	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	R	SP
Forbidden Triples	LO	NPA	PS	LFI	EM	R	LR	DLF	RN	UPR	SP

- SP R UPR are "*small*" candidates that are **hard to classify**.

French presidential elections

We can compare to a **baseline**: **axes used by poll institutes**.

BVA	LO	NPA	LFI	PS	EM	LR	DLF	RN	UPR	SP	R
IFOP	LO	NPA	LFI	PS	EM	R	LR	DLF	RN	UPR	SP
IPSOS	LO	NPA	LFI	PS	EM	R	LR	DLF	RN	SP	UPR
Harris	LO	NPA	LFI	PS	EM	R	LR	DLF	SP	UPR	RN
Elabe	LO	NPA	LFI	PS	EM	R	LR	UPR	DLF	RN	SP

French presidential elections

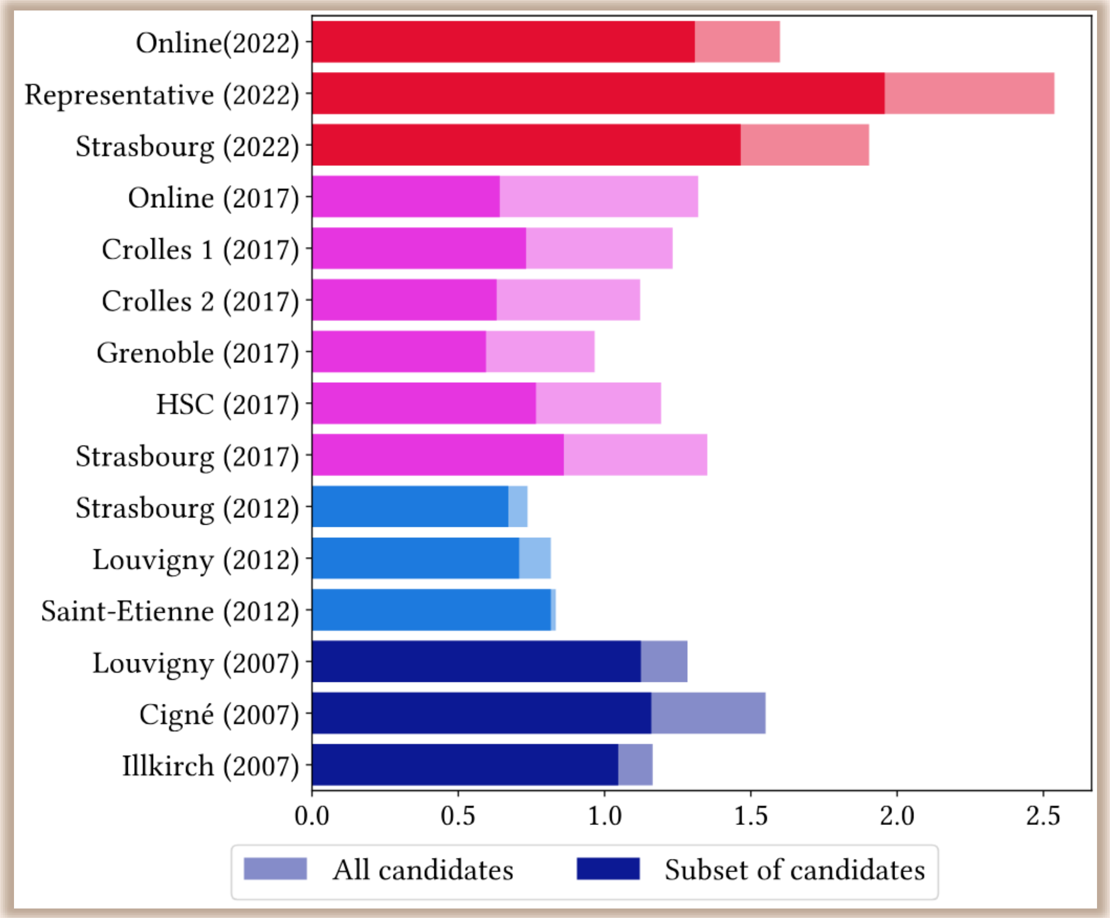


Fig. Maximal Swap costs (per voter) of the optimal axes with all candidates and when we remove "*small*" candidates that are hard to classify.

2022: 1 candidate.
2017: 3 candidates.
2012: 1 candidate.
2007: 2 candidates.

Supreme Court of the US

For each case, justices can **join opinions** (*majority opinion*, *dissenting opinion*, *concurring opinion*)

One opinion = One ballot

- **65 terms** between 1946 and 2021, with $m = 9$ justices in average
  $n = 240$ opinions per term.

Supreme Court of the US

We compare our results to the baseline, called **Martin-Quin method**, based on a Bayesian method to assign position to justices based on the majority votes.

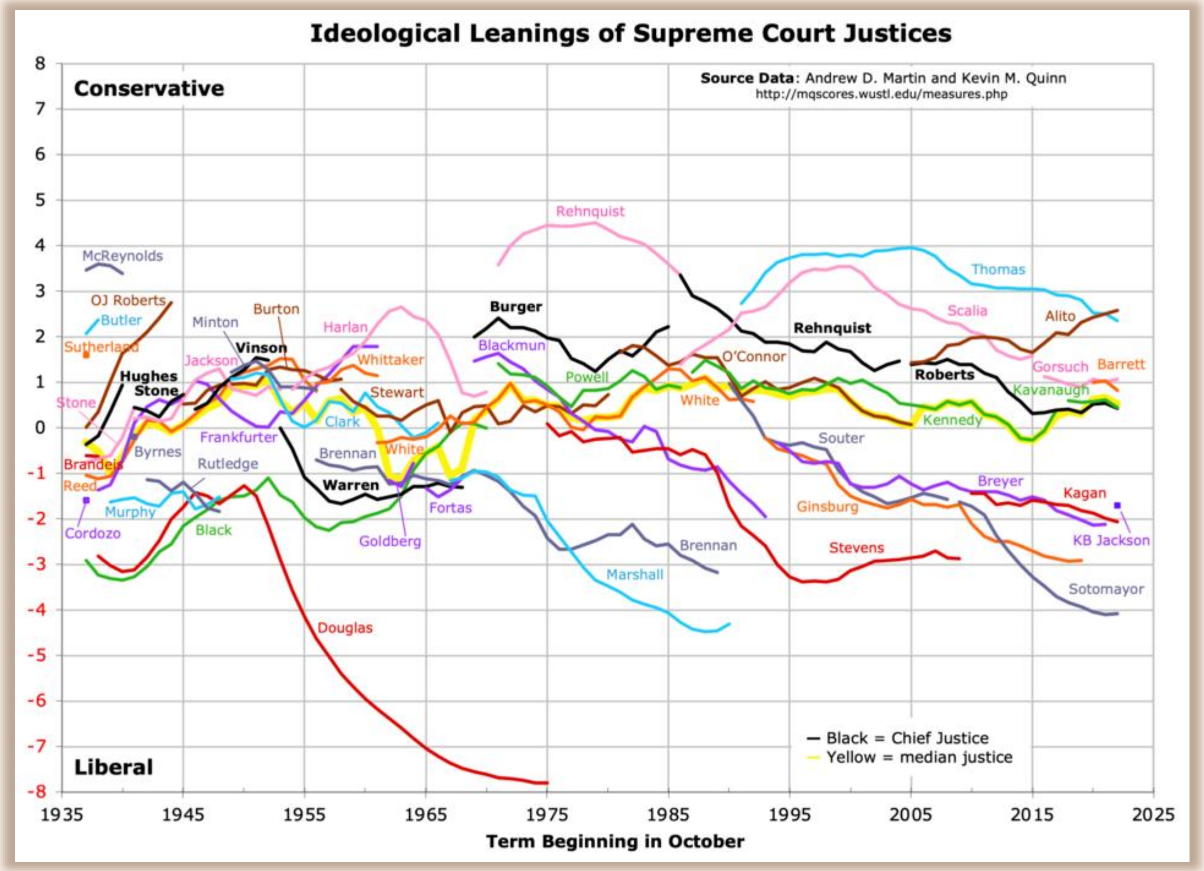


Fig. Evolution of the MQ positions of the justices through time.

Supreme Court of the US

Rule	Avg KT	Same median	Same axis
VD	4.94	53.8 %	1.54 %
MF	4.22	58.5 %	3.08 %
BC	3.68	56.9 %	3.08 %
MS	3.55	64.6 %	1.54 %
FT	**3.43**	**66.2 %**	**7.69 %**

Fig. Comparison between the axes returned by our rules and the Martin-Quinn axes.

Supreme Court of the US

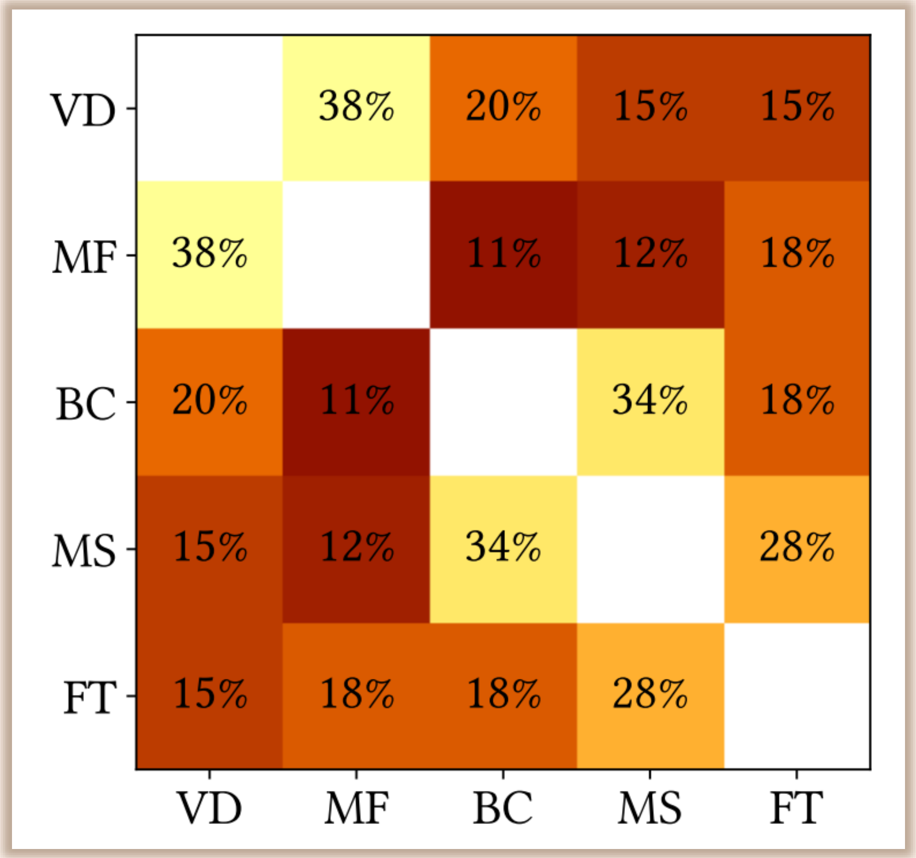


Fig. % of terms for which axes returned by different rules are perfectly matching.

Supreme Court of the US

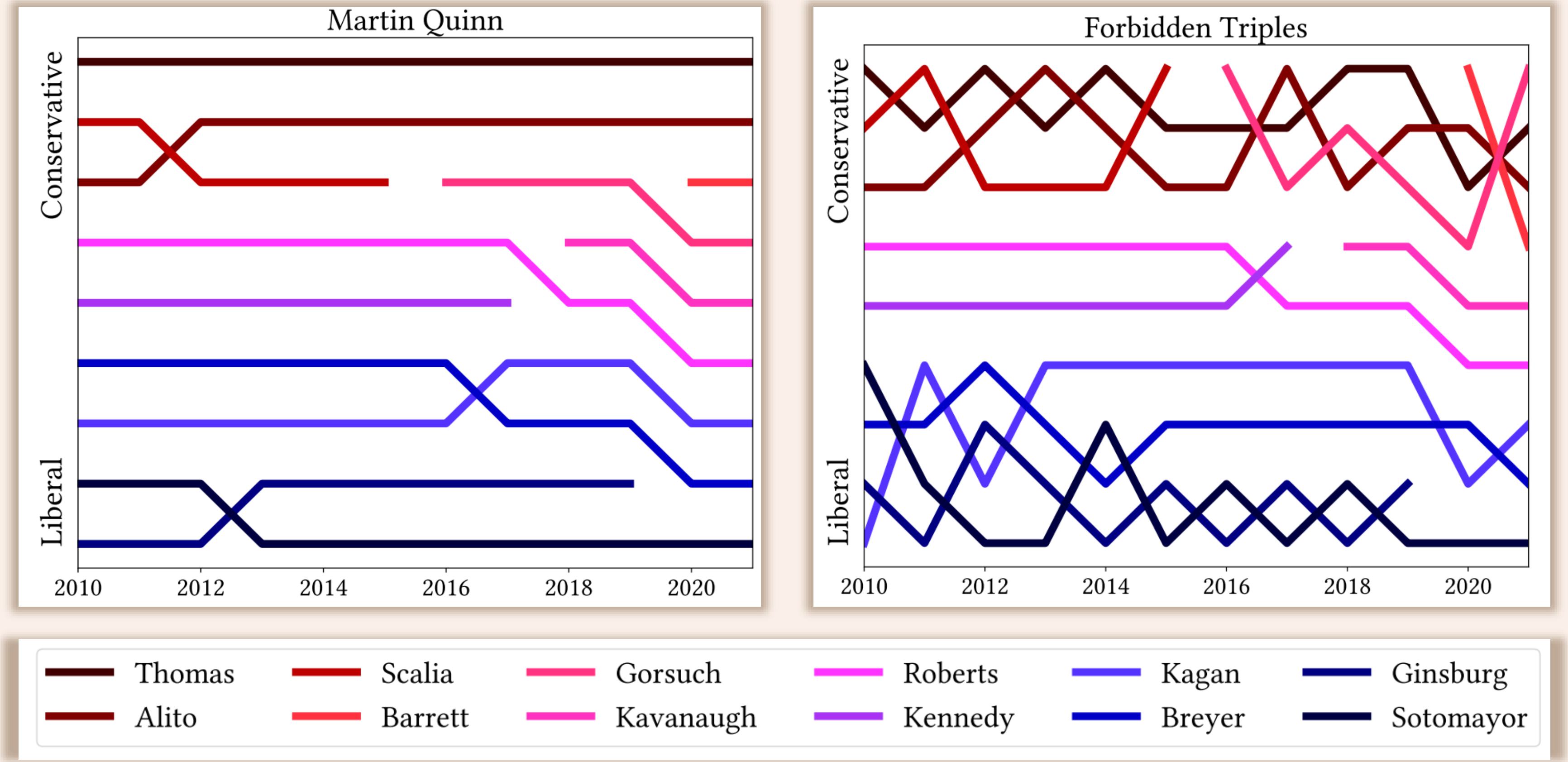
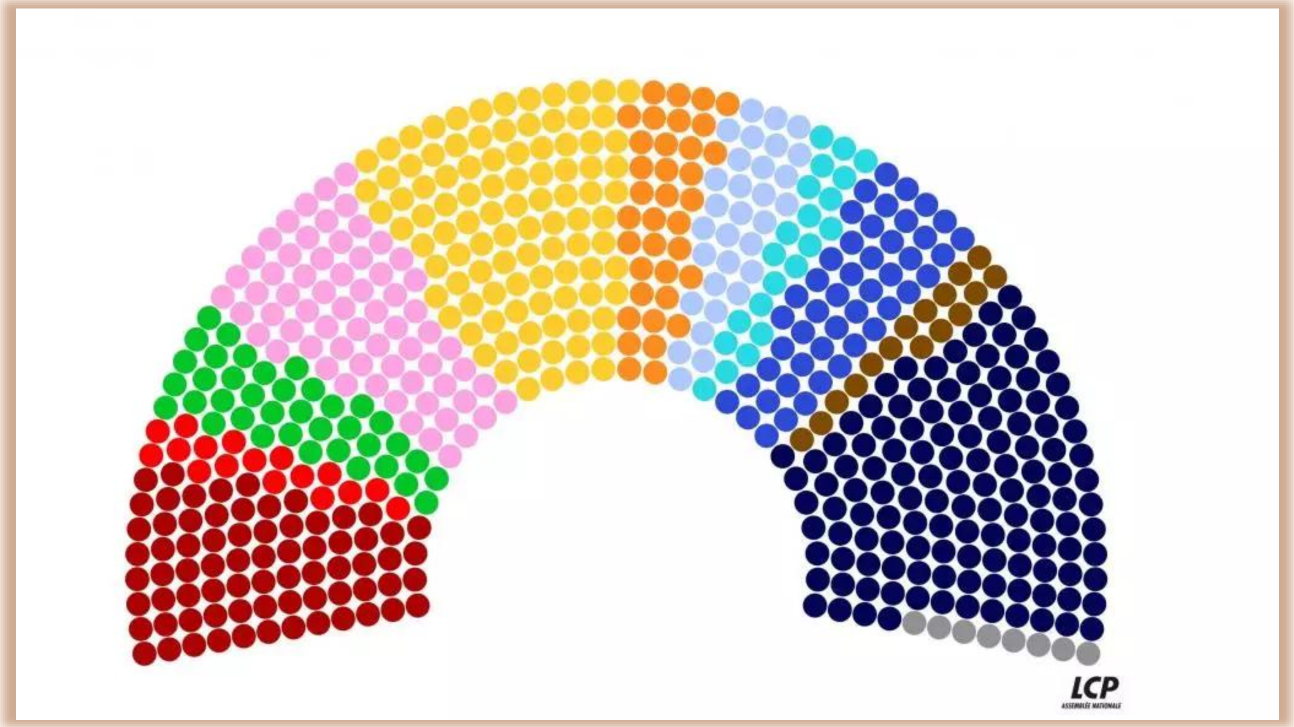


Fig. Evolution of the axes for successive terms (MQ and FT).

Applauses in French parliament



How can we re-construct the left-right order of **the French Parliament**?

Applauses in French parliament

M. Laurent Nuñez Gouv
(ministre)

L'ordre a été maintenu.
(Applaudissements sur plusieurs bancs des groupes RN , EPR *et* HOR .)

↓

A ballot approving **RN**, **EPR** and **HOR**.

- We have $n = 4842$ applauses for $m = 10$ groups.
- See the interactive tool

Tierlists: pop-culture

Fig. Tierlist of Star Wars movies.

Tierlists: pop-culture

S	Approval	Approval IRV	Approval with runoff
A	Borda	IRV	Range voting
B	Ranked Pairs	Majority Judgement	
C	Plurality with Runoff		
D	Plurality	Split approval	Veto
F	Dictatorship		

Fig. Tierlist of single-winner voting rules.

Tierlists: pop-culture

Dataset	n	m	m^*	#app.
Planets	58	9	–	2.64
Numbers	50	10	–	2.56
Months	244	12	–	2.80
Harry Potter Movies	324	8	–	2.66
Star Wars Movies	4 002	11	–	2.71
Spider Man Movies	346	10	–	2.80
Taylor Swift Albums	169	12	–	3.24
Colors	618	11	–	3.08
School Courses	214	11	15	2.67
European Countries	624	10	51	3.94

Table. Datasets of tierlist collected from *Tiermaker.com*.

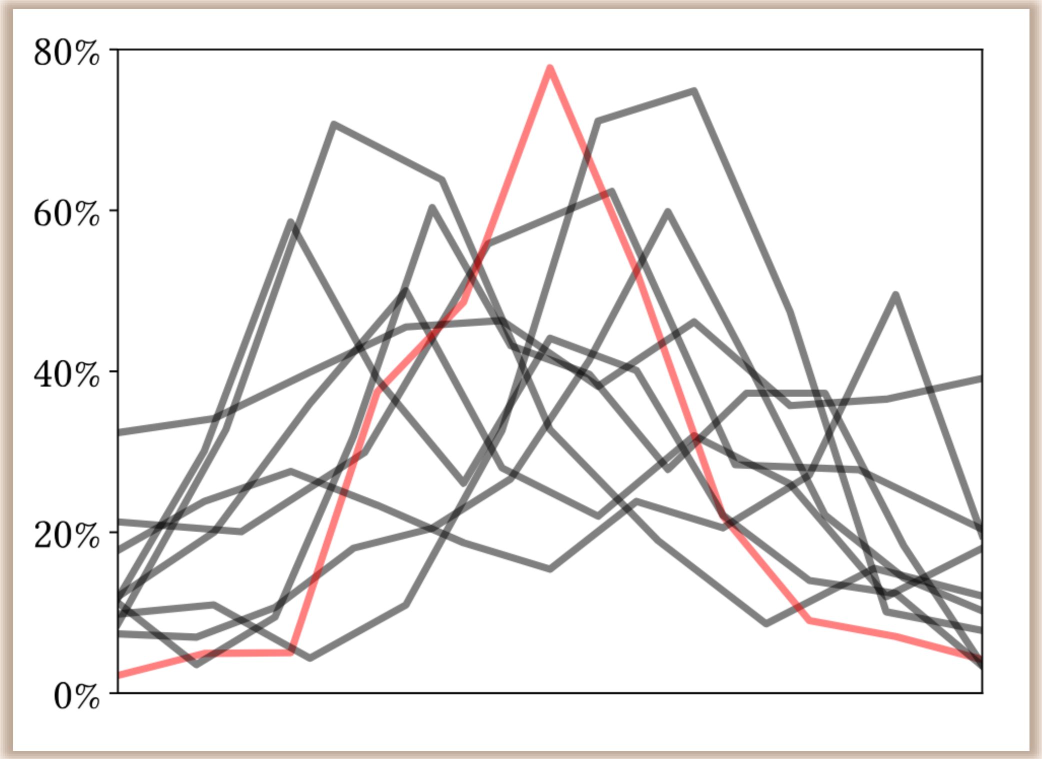
The approval ballots are the set of candidates ranked in the top category by the voter.

Tierlists: pop-culture

Dataset	Months	Star Wars	Spider Man	School Courses	European Countries
	March	Solo	Venom	Chemistry	Italy
	Apr.	Ep. II	Amazing SM 2	Physics	France
	May	Ep. I	Amazing SM 1	Math	UK
	Aug.	Rogue One	SM 3	Technology	Germany
	June	Ep. III	SM 1	Music	Switzerland
Axis ◁	July	Ep. V	SM 2	Art	Sweden
	Dec.	Ep. IV	Spiderverse	PE	Norway
	Oct.	Ep. VI	No way home	History	Denmark
	Nov.	Ep. VIII	Homecoming	Social studies	Iceland
	Sep.	Ep. VII	Far from home	Foreign lang.	Finland
	Jan.	EP. IX		Literature	
	Feb.				

Fig. Axes returned by Forbidden Triples. Colors indicate groups of candidates (e.g., seasons or trilogy).

Tierlists: pop-culture



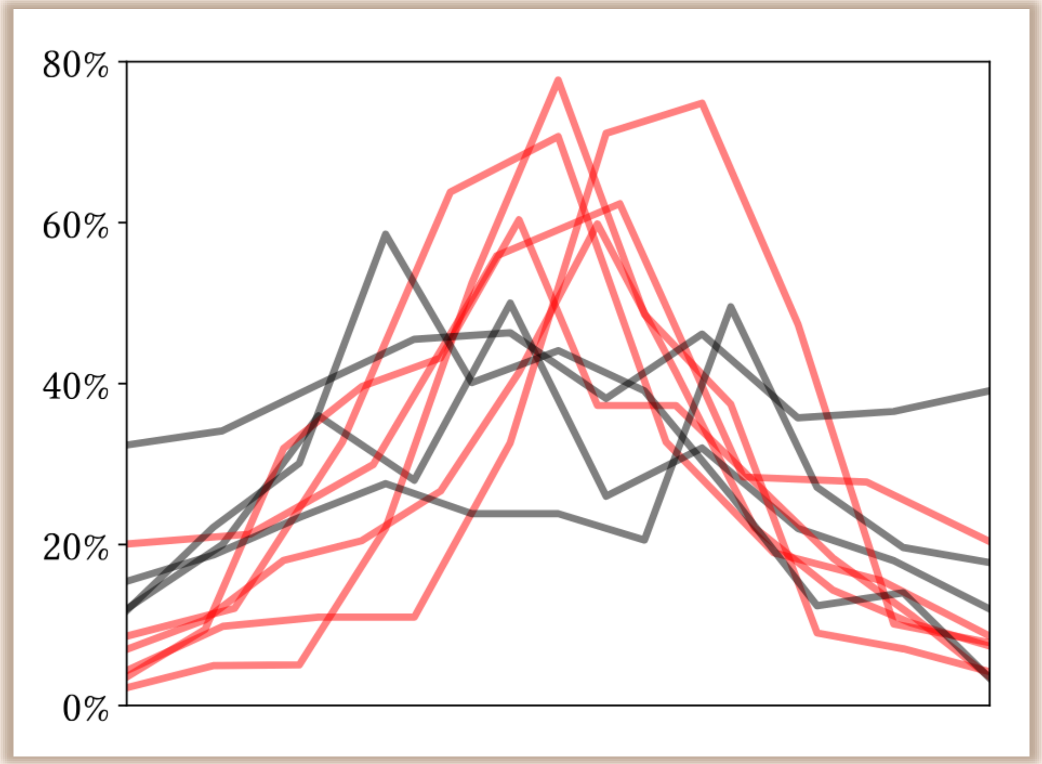
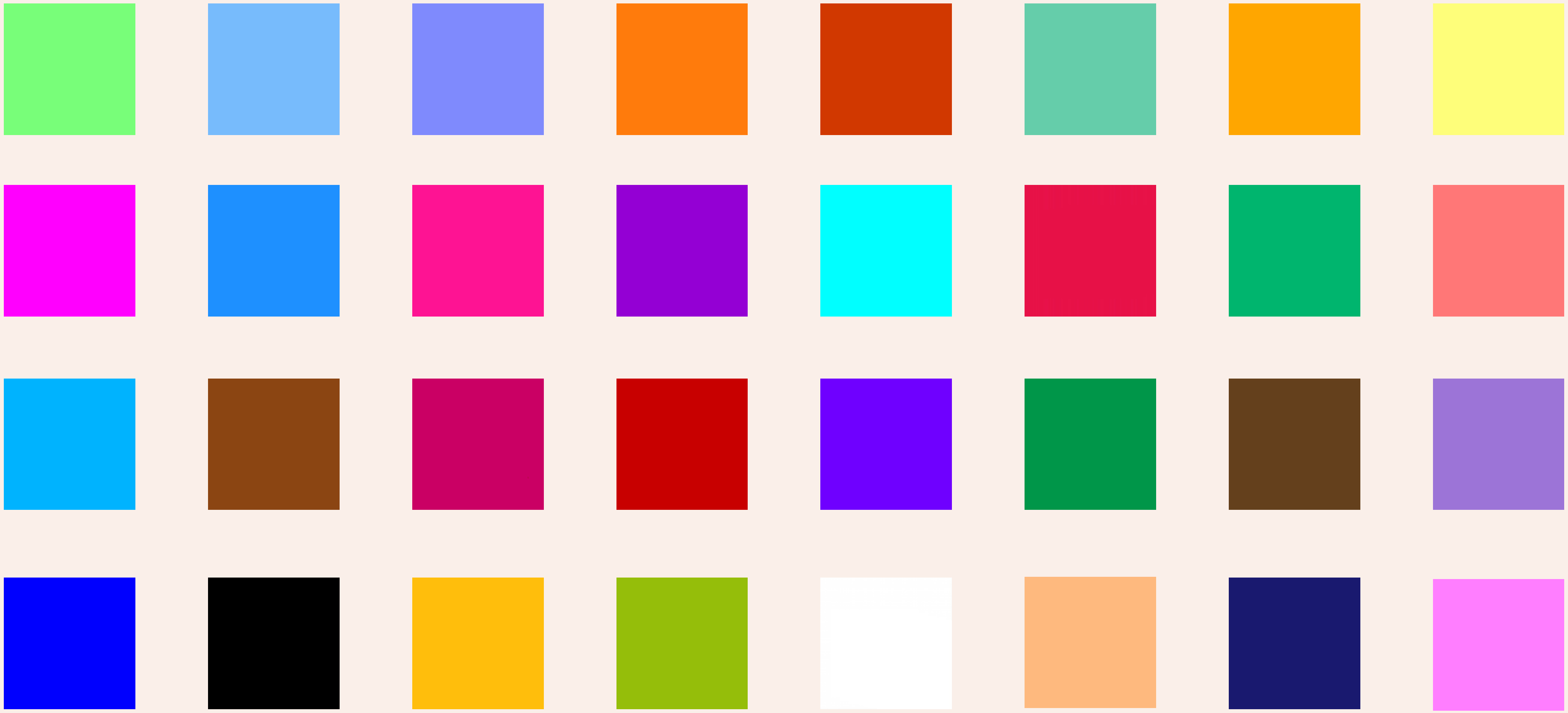
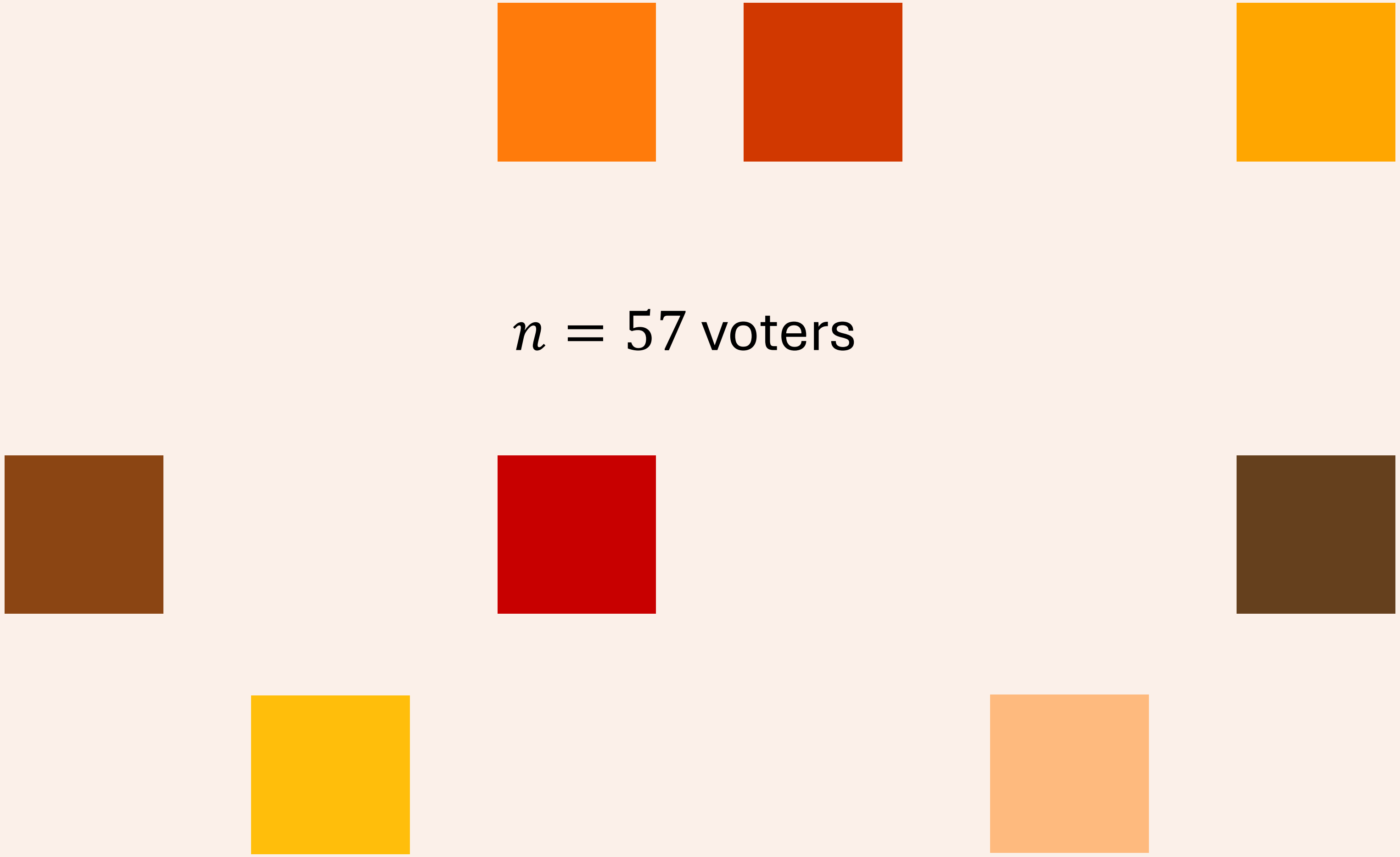


Fig. Distribution of the approval scores of candidates **along the order of the axis** returned by VD (left) or FT (right). Red indicate single-peaked distributions.

Tierlists: colors



Tierlists: colors



$n = 57$ voters

Tierlists: colors

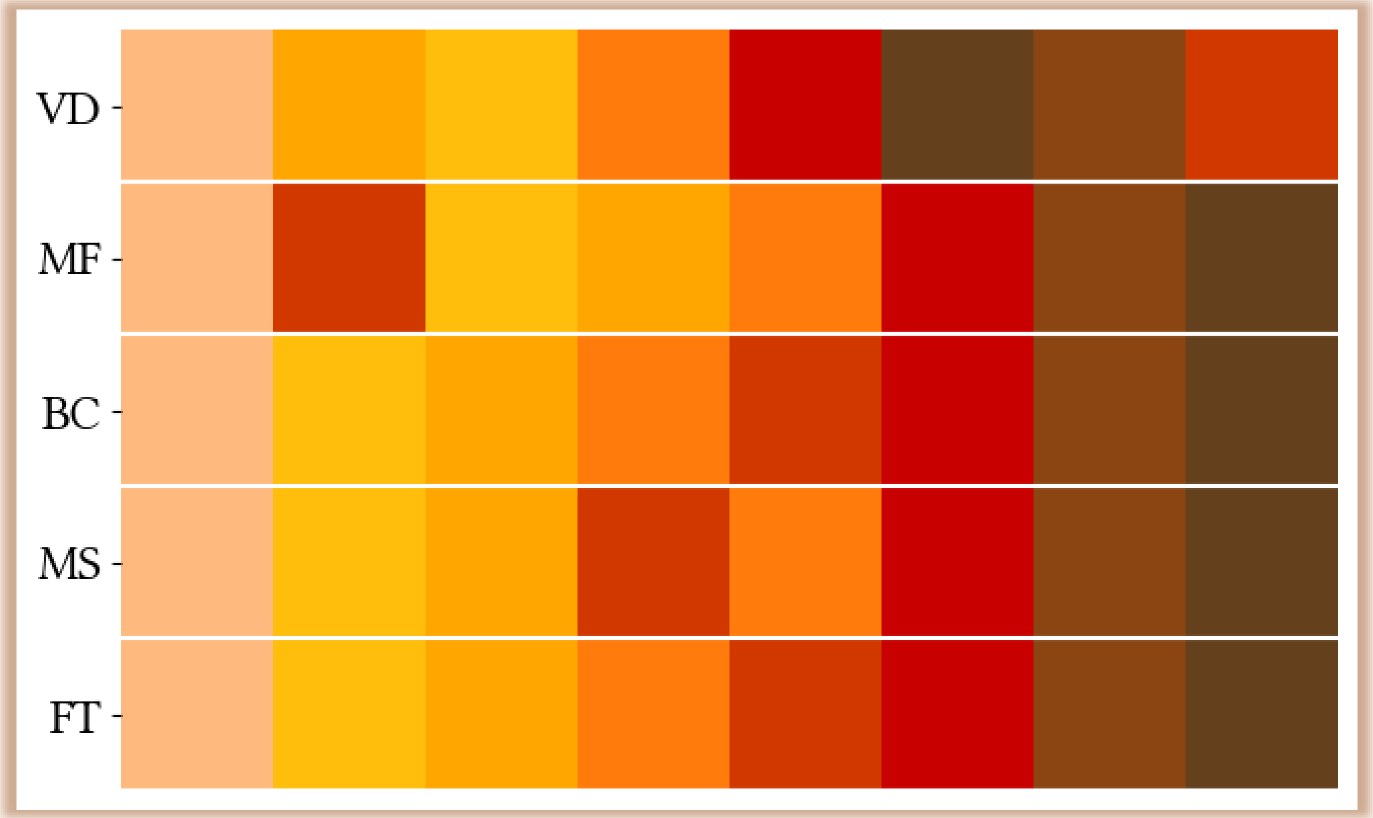


Fig. Colors axes returned by the axis rules.

Tierlists: colors

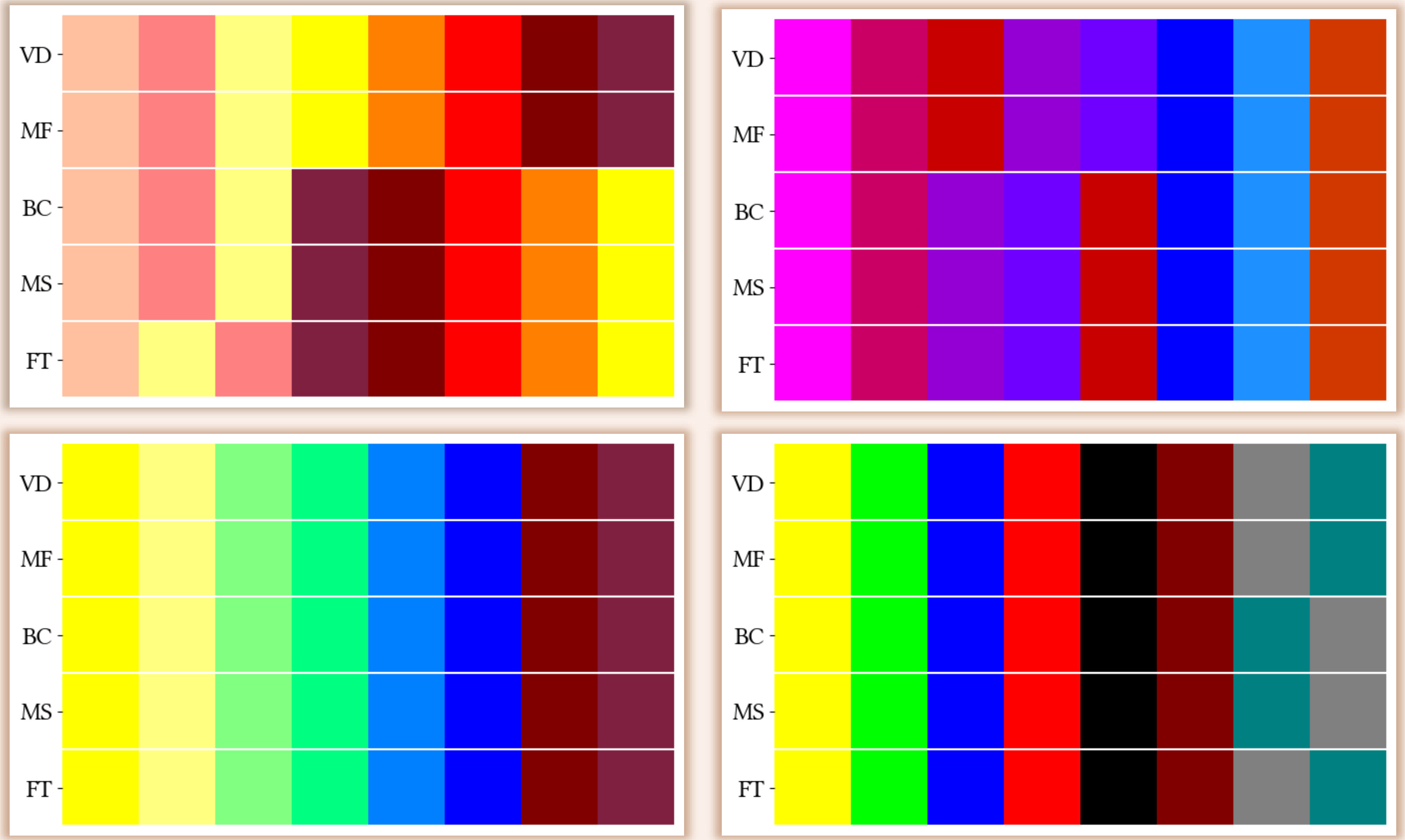


Fig. Colors axes returned by the axis rules.

Experiments: **conclusion**

- The rules can correctly identify **clusters** of similar candidates, and return axes **close to the "natural" axes**.
- The **more information** the rule uses (e.g., FT uses much more than VD), the better it seems to perform.
- Unpopular or unknown candidates are **pushed towards the extremes** while popular candidates are **pushed towards the center**.

Conclusion and Extensions

Conclusion: Take-away

Rules can be ordered on a spectrum from the **least informational** (VD) to the **most informational** (FT). Rules that use more information have more advantages **in terms of axis quality**, and the rules that use less information are **easier to interpret**. Ultimately, the choice of the rule should depend **on the context** of the application.

Extensions

- **Greedy variants** of the rule for fast computation and handling more candidates.
- Variants for **circular axes** (horseshoe hypothesis).
- **Time-consistent** variants of the rule (e.g., axis for year t depends on the one from year $t-1$).
- Variants considering the **popularity of candidates.**
- Variants for **incomplete information** (approved, disapprove or *unknown*).

Thanks for your attention!

Read the full paper.

theo.delemazure.fr/more/candidate-ordering.pdf

Read it in my thesis.

theo.delemazure.fr/storage/thesis-theo-delemazure.pdf