Independence of Irrelevant Alternatives
Under the Lens of Pairwise Distortion

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Context: Voting

Input: A group of n voters have ranking preferences over m candidates.

$$A \succ B \succ C \succ D \succ E \succ F$$

$$E \succ F \succ B \succ C \succ D \succ A$$

$$B \succ E \succ D \succ A \succ F \succ C$$

$$A \succ F \succ D \succ E \succ C \succ B$$

Output: A ranking of the m candidates.

$$A \succ B \succ E \succ F \succ D \succ C$$

Arrow's Impossibility Theorem (1951)

For $m \geq 3$ candidates, if a voting rule satisfies
(1) Unrestricted domain,
(2) Pareto-efficiency and
(3) **Independence of Irrelevant Alternatives**,
then it is Dictatorship.

Interpretation 1

There is **no good voting rule**.

Interpretation 2

IIA is **too strong**.

Arrow's Impossibility Theorem (1951)

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Interpretation 1

There is **no good voting rule**.

Interpretation 2

IIA is **too strong**.

Hypothesis

It is **not necessarily too bad** to fail IIA.

The **IIA** property

Independence of Irrelevant Alternatives

The collective choice between two candidates is independent of the positions of all other candidates.

$A \succ B \succ C \succ D \succ E \succ F$

$E \succ F \succ B \succ C \succ D \succ A$

$B \succ E \succ D \succ A \succ F \succ C$

$A \succ F \succ D \succ E \succ C \succ B$

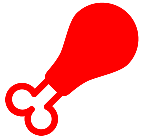
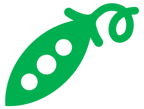
→ Prevent spoiler effect and vote splitting.

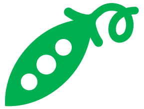
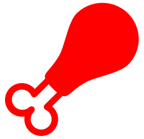
→ Prevent the implicit use of interpersonal comparisons.

Example

55%	$A \succ B \succ C \succ D \succ E \succ F$
45%	$B \succ E \succ F \succ C \succ D \succ A$

Example

55%  $\succ$  $\succ$ **C** $\succ$ **D** $\succ$ **E** $\succ$ **F**

45%  $\succ$ **E** $\succ$ **F** $\succ$ **C** $\succ$ **D** $\succ$ 

Our goal

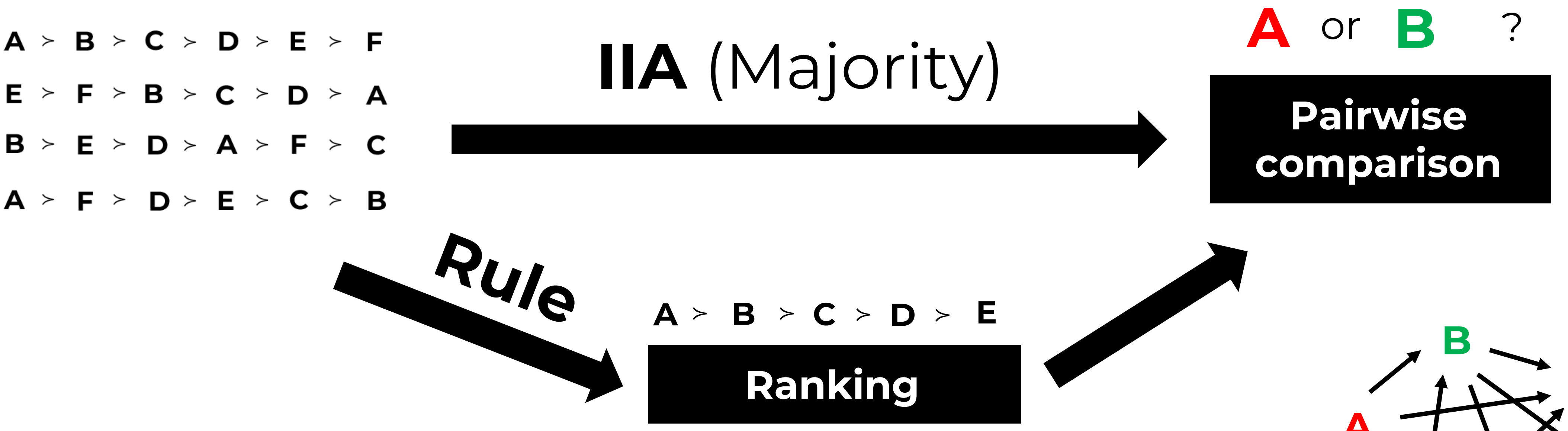
Measuring the *(negative)* impact of **IIA** on social welfare.

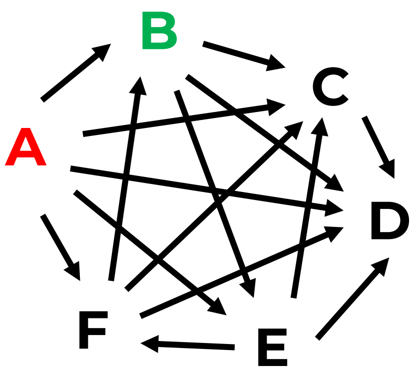
Our goal

Measuring the *(positive)* impact of **additional candidates** on social welfare.

A common **model**

We need a model to compare IIA and Voting Rules.

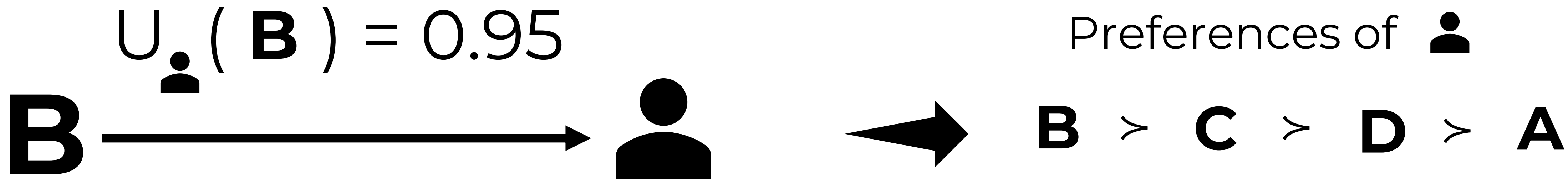




We use the model of **Pairwise Voting Rules**.

Distortion (Procaccia and Rosenschein, 2006)

Voters have **utilities** over candidates, on which are based the preferences.

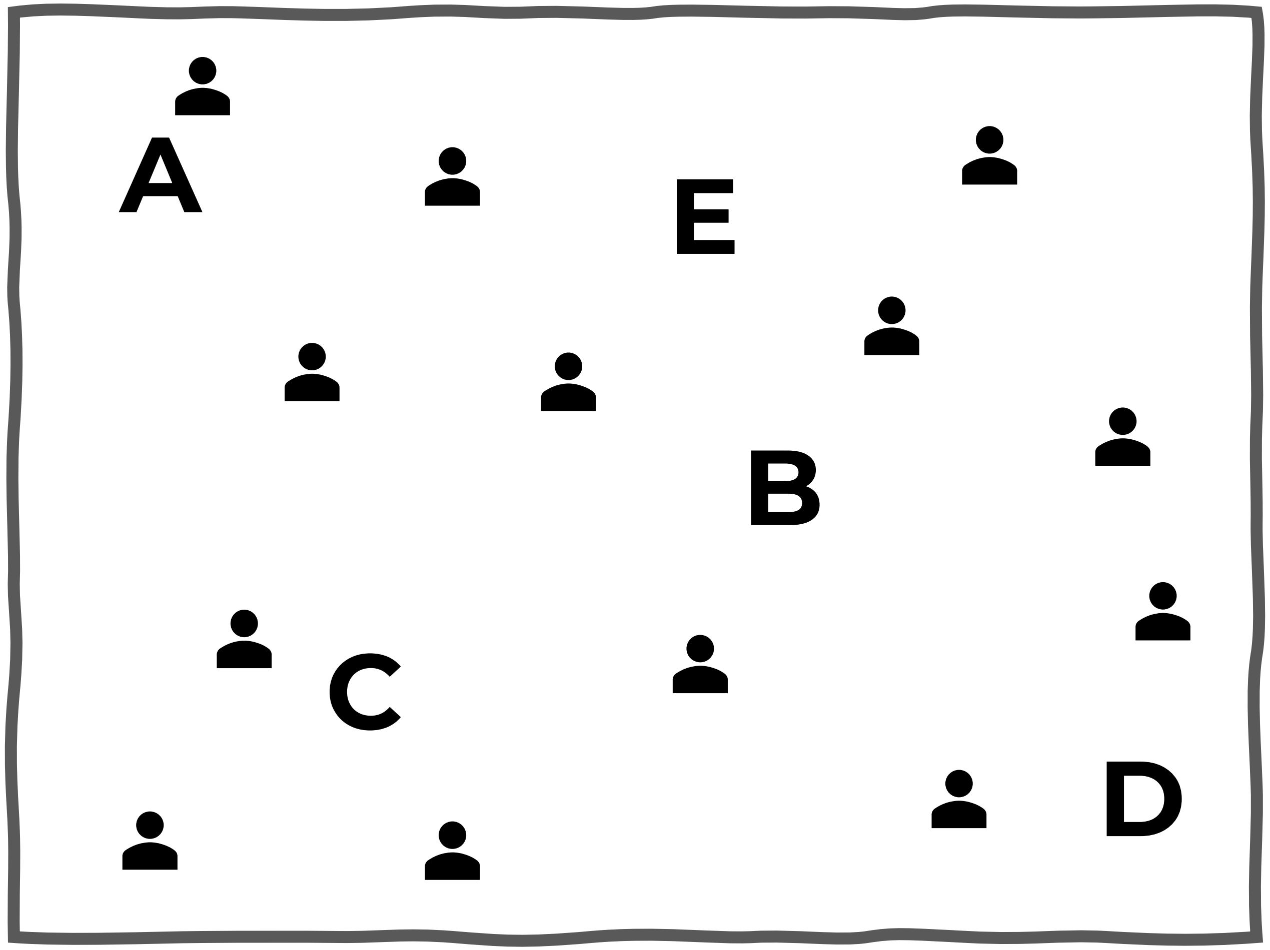


The **distortion of a rule** is defined as the ratio:

$$\frac{\textbf{Total utility of the optimal candidate}}{\textbf{Total utility of the candidate selected by the rule}}$$

Metric Distortion (Anshelevich et al. 2018)

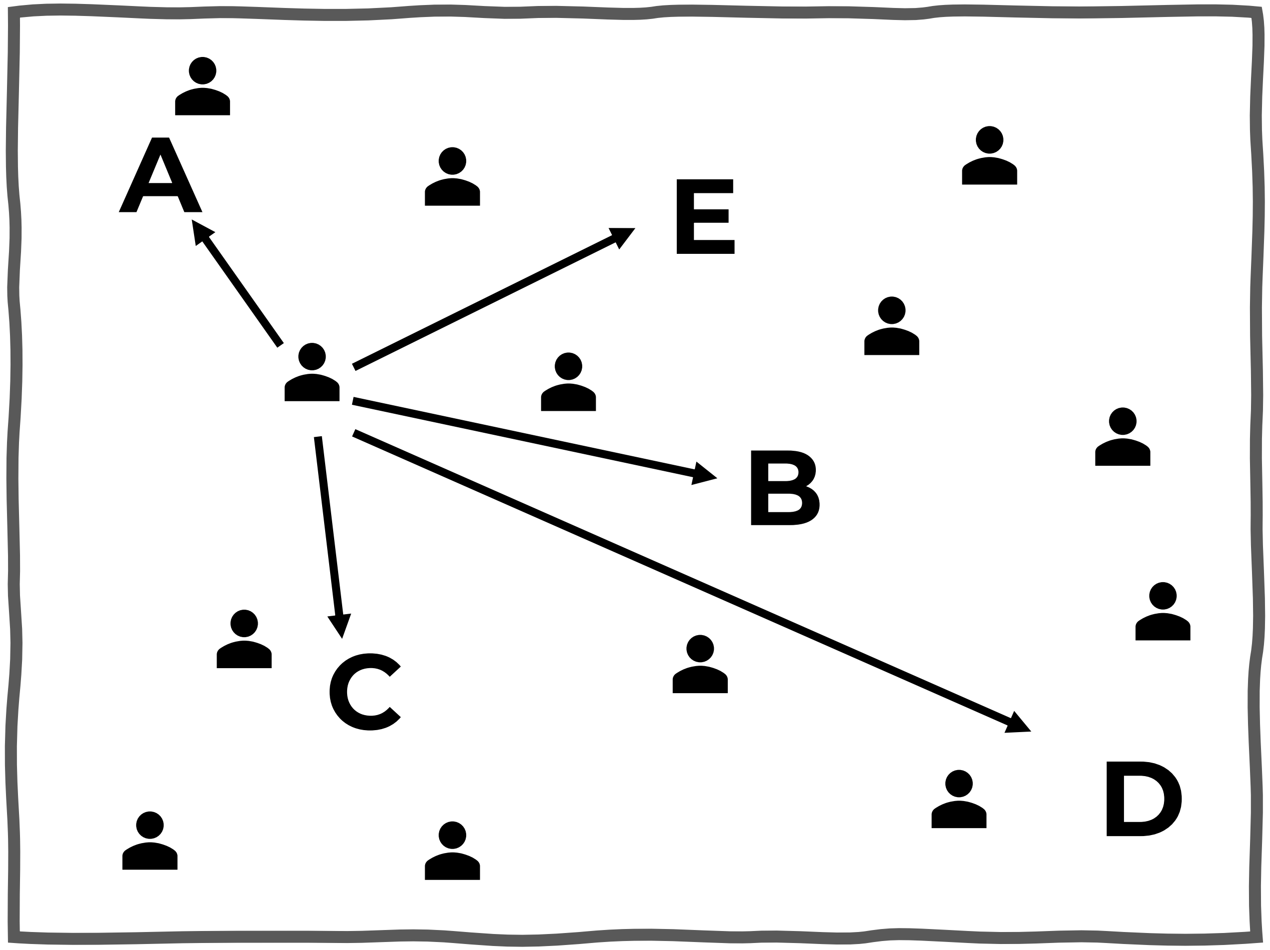
A metric space.



Metric Distortion (Anshelevich et al. 2018)

A metric space.

The distance induces a cost on which preferences are based.



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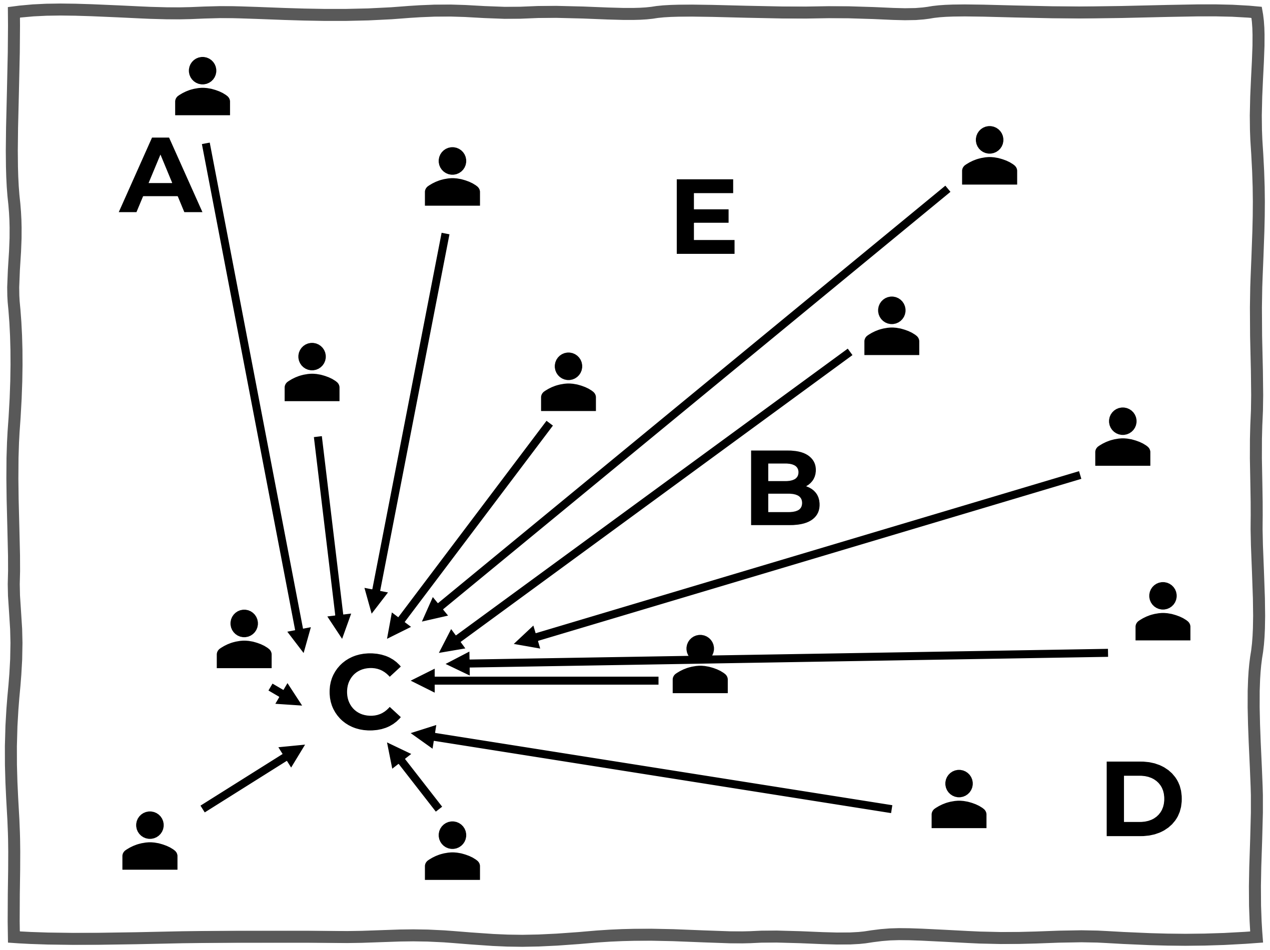
This gives **a total cost** for each candidate.



Metric Distortion (Anshelevich et al. 2018)

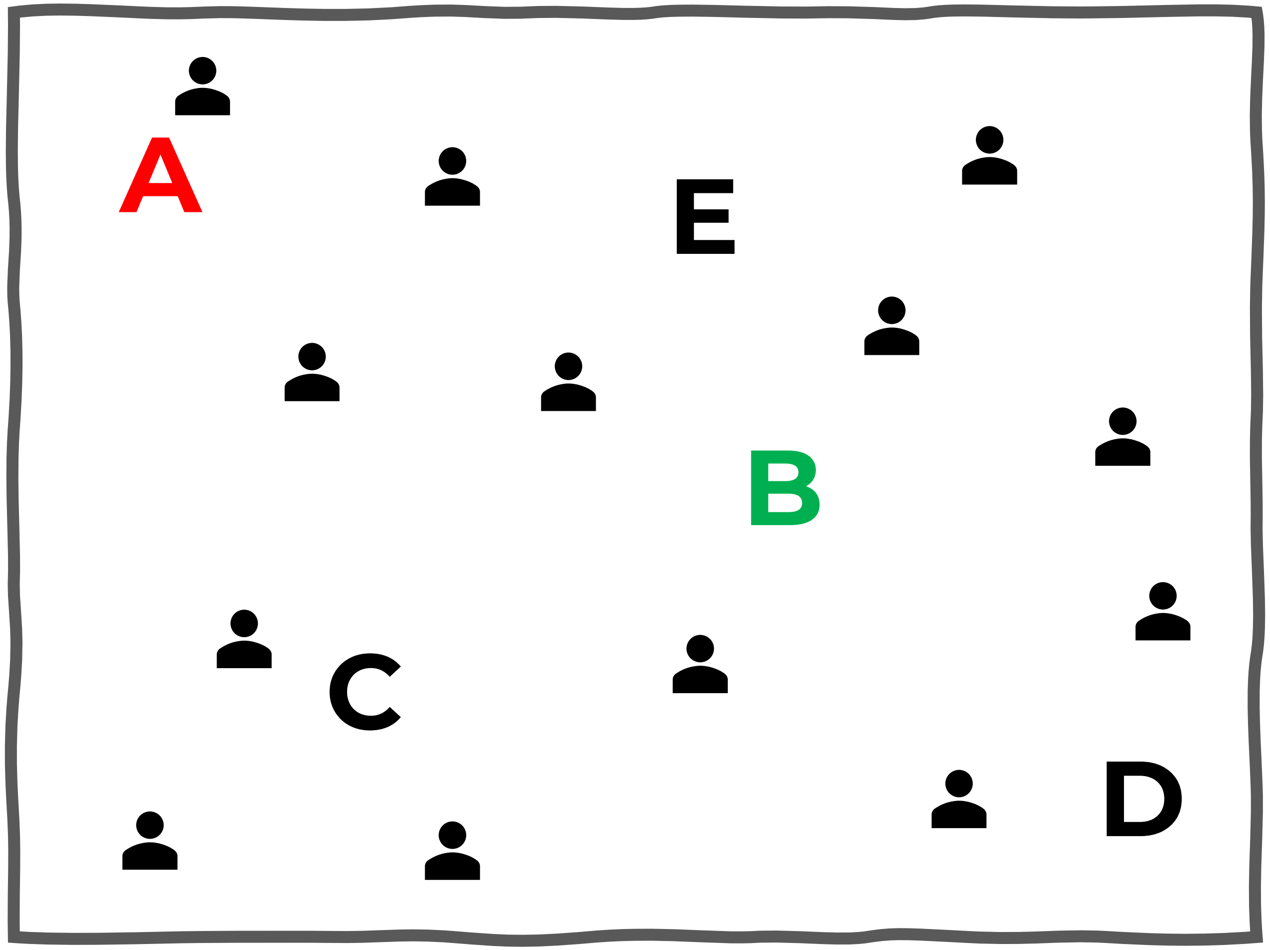
A metric space.

The distance induces a cost on which preferences are based.

This gives **a total cost** for each candidate.

The distortion is the ratio:

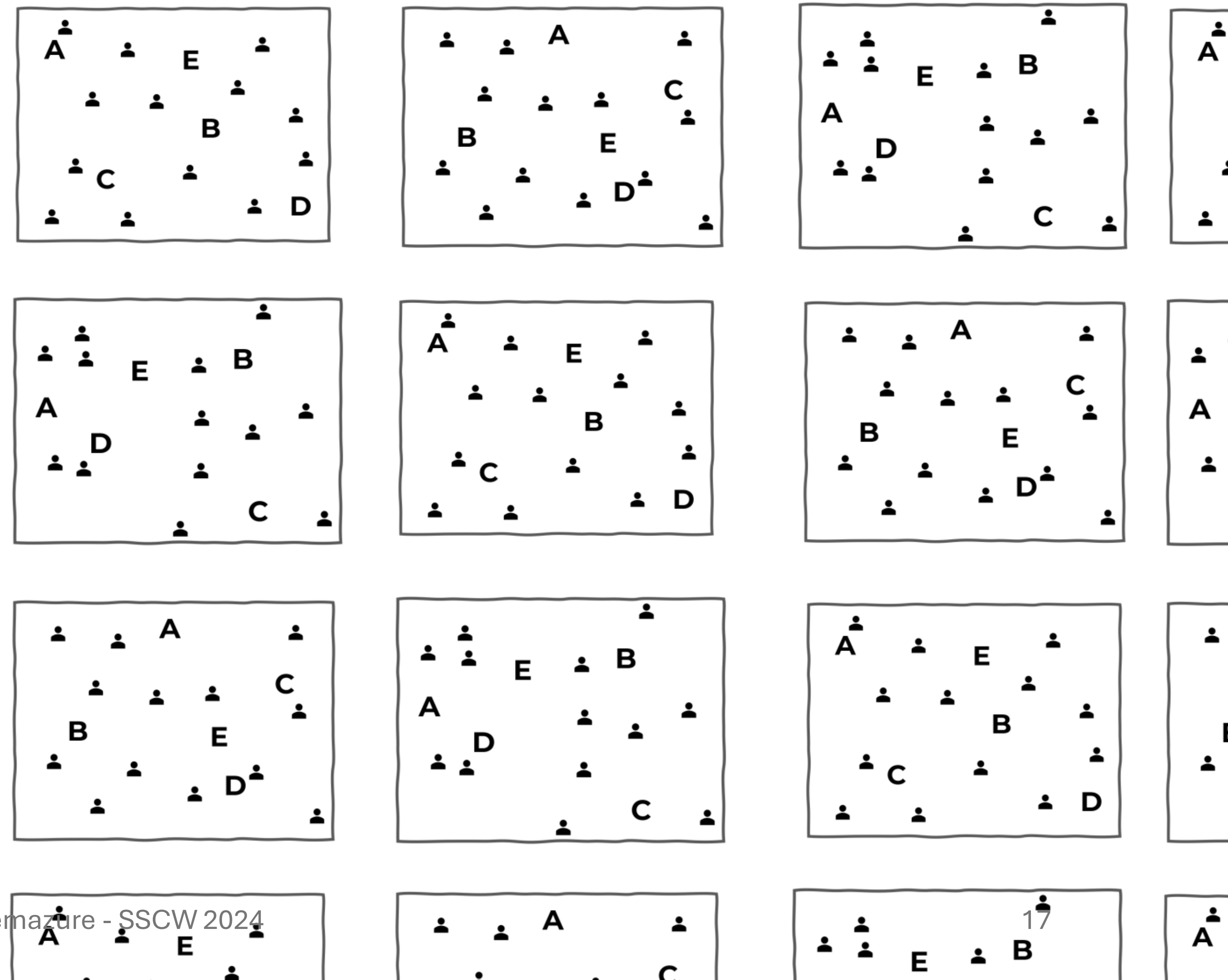
$$\frac{\textbf{Cost of the candidate } \textbf{selected by the rule}}{\textbf{Cost of the } \textbf{optimal candidate}}$$



Average Case Analysis

Average Pairwise Distortion

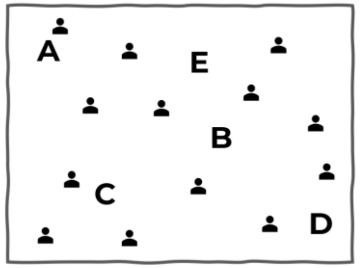
1. **Sample random** utilities/positions in the metric space.



Average Pairwise Distortion

1. **Sample random** utilities/positions in the metric space.

2. **For each pair of candidates**, compute the pairwise distortion of the rule.



dist(**A**,**B**) = **1**　*dist*(**A**,**C**) = **1.2**　*dist*(**A**,**D**) = **1**

dist(**A**,**E**) = **1**　*dist*(**B**,**C**) = **1**　*dist*(**B**,**D**) = **1.5**

dist(**B**,**E**) = **1.3**　*dist*(**C**,**D**) = **1**　*dist*(**C**,**E**) = **1.1**

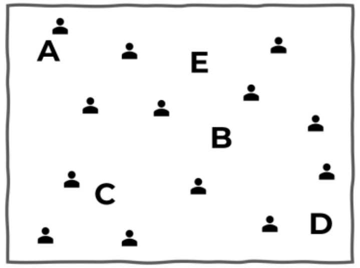
dist(**D**,**E**) = **1**

Average Pairwise Distortion

1. **Sample random** utilities/positions in the metric space.

2. **For each pair of candidates**, compute the pairwise distortion of the rule.

3. Compute the **average** of all pairwise distortions.



dist(**A**,**B**) = **1**
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dist(**B**,**E**) = **1.3**
dist(**C**,**D**) = **1**
dist(**C**,**E**) = **1.1**
dist(**D**,**E**) = **1**

avg-dist = **1.11**

Average Pairwise Distortion

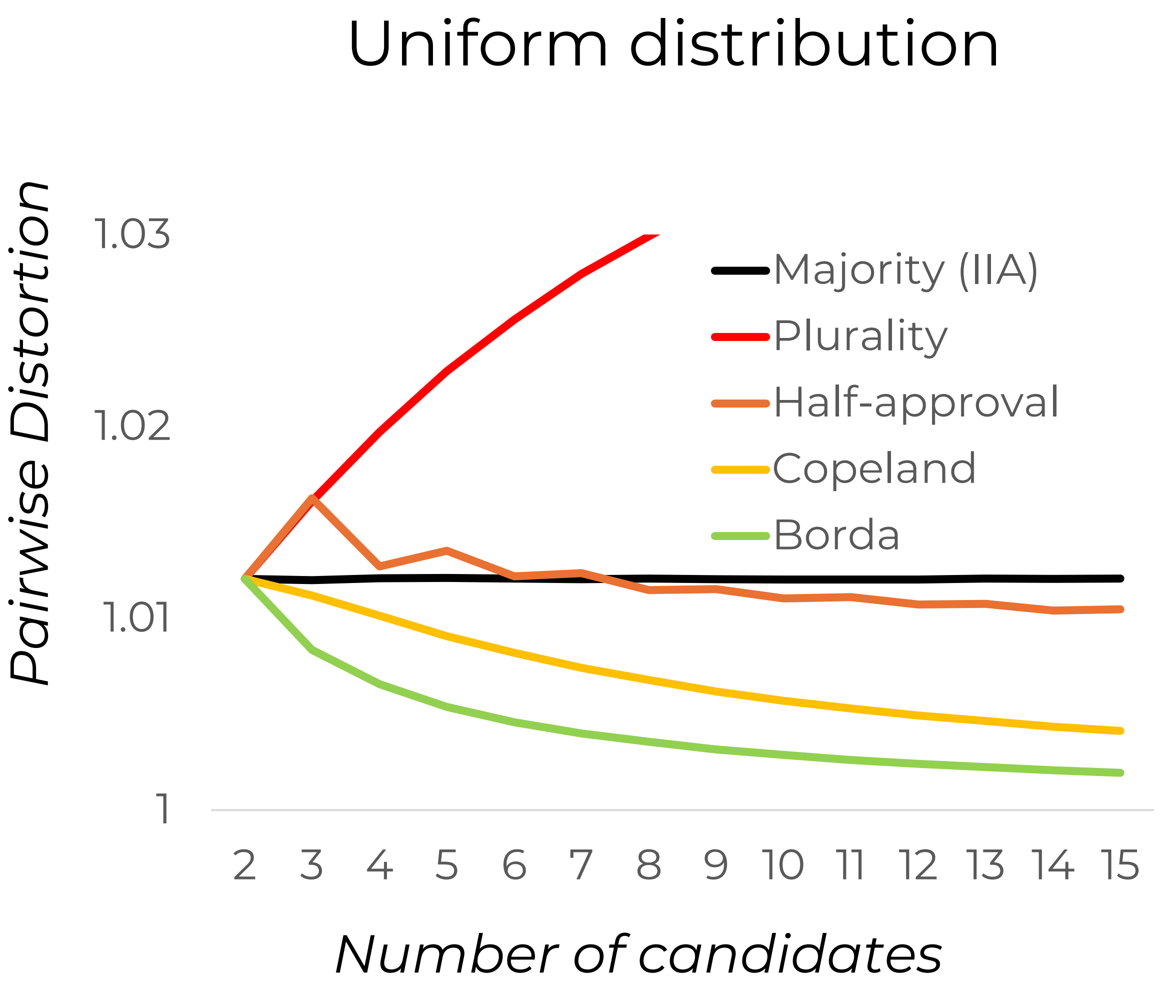
Costs Datasets:

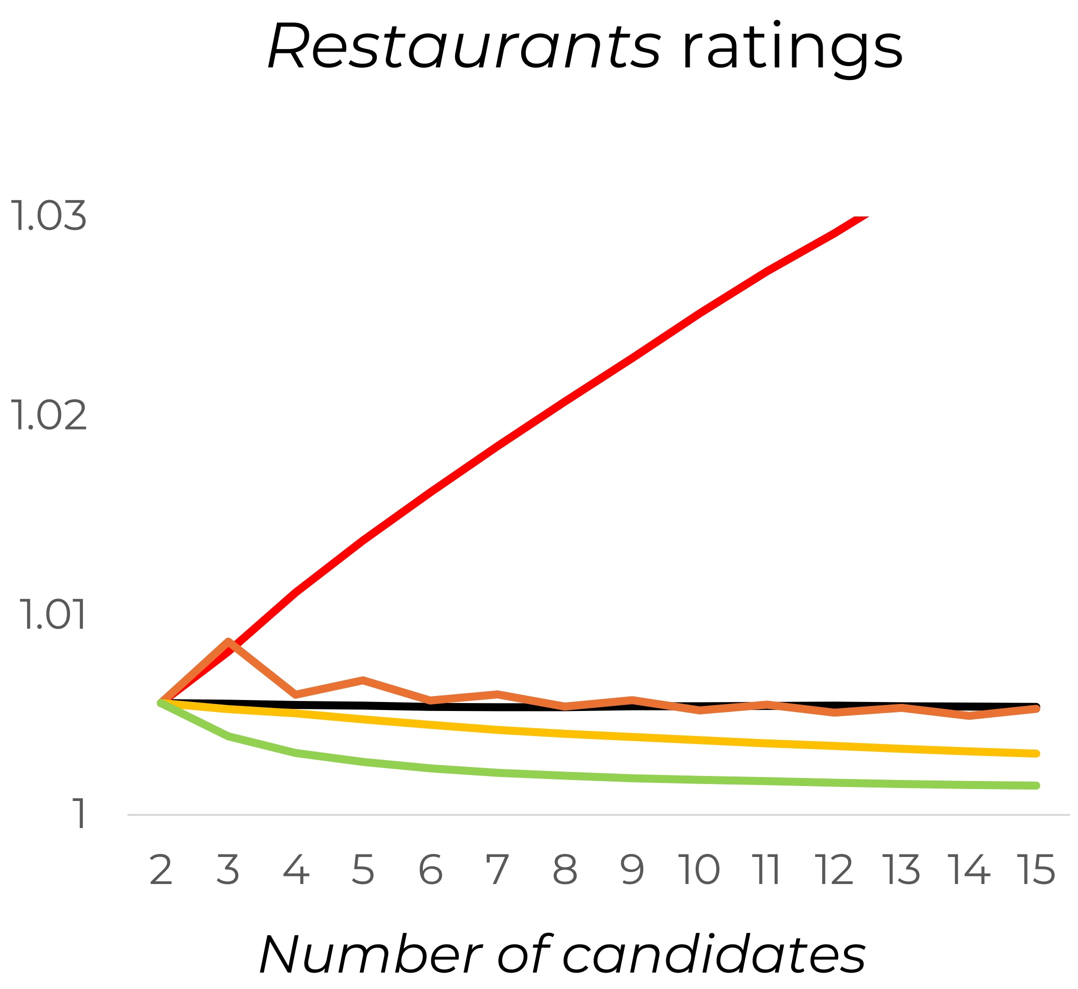
- 1D, 2D, 3D and 10D metric space
- Euclidean and Manhattan distances

Utilities Datasets:

- Uniform distribution in [0,1]
- Normalized uniform distribution (sum of utilities = 1 for each voter)
- Gaussian distribution N(0.5,0.5)
- Gaussian distribution with uniform distribution of center for each voters
- Based on datasets of ratings of bars and restaurants

Average Pairwise Distortion





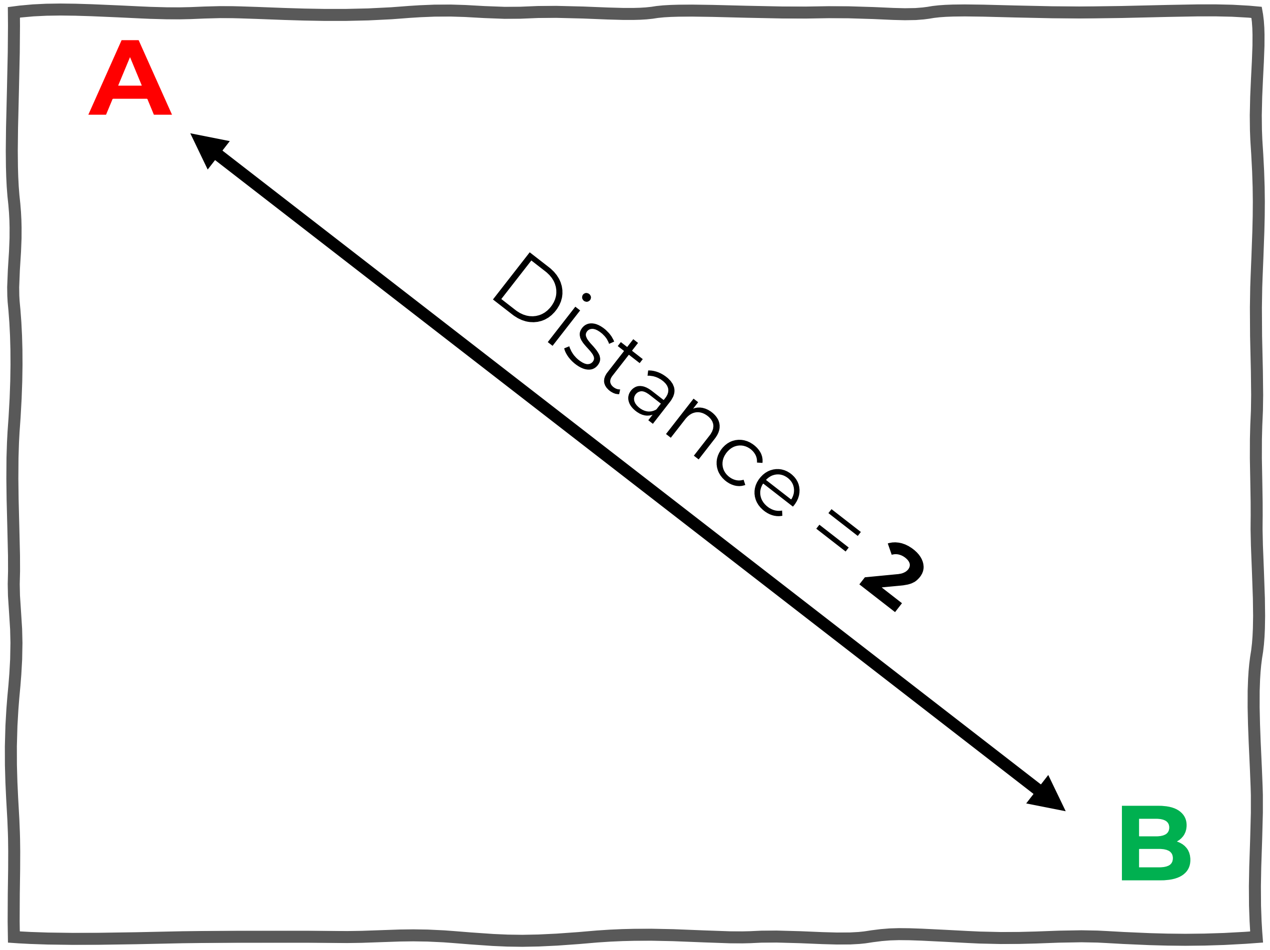
Worst Case Analysis

Worst-case Pairwise Distortion

Metric space.

2 candidates.

What is the worst-case pairwise distortion of **Majority vote**?



Worst-case Pairwise Distortion

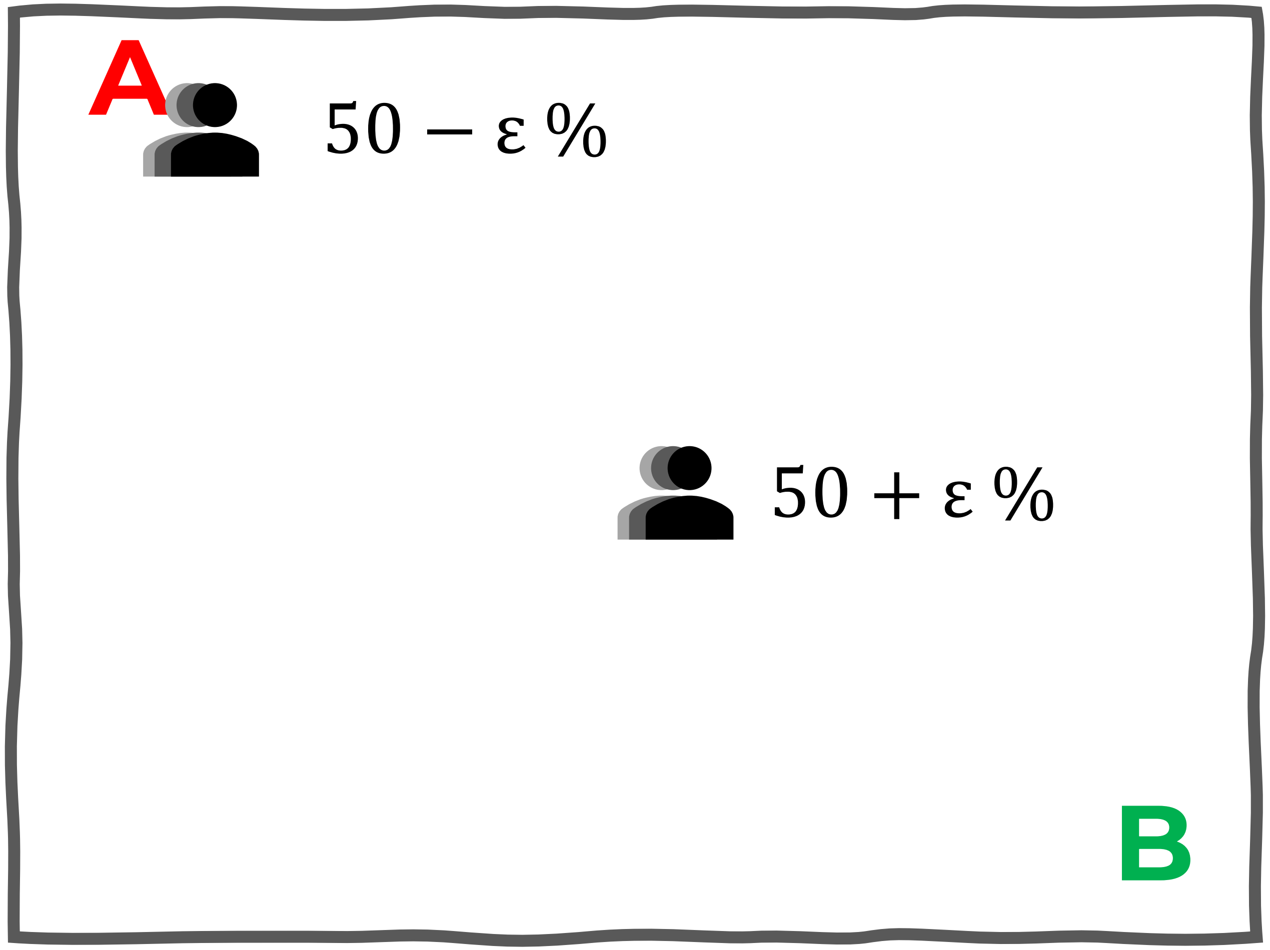
Distortion of **A** and **B**

$50 - \varepsilon$ **A** $\succ$ **B**

$50 + \varepsilon$ **B** $\succ$ **A**

Worst-case distortion of
Majority vote:

$$\frac{0.5 \times 2 + 0.5 \times 1}{0.5 \times 0 + 0.5 \times 1} = 3$$



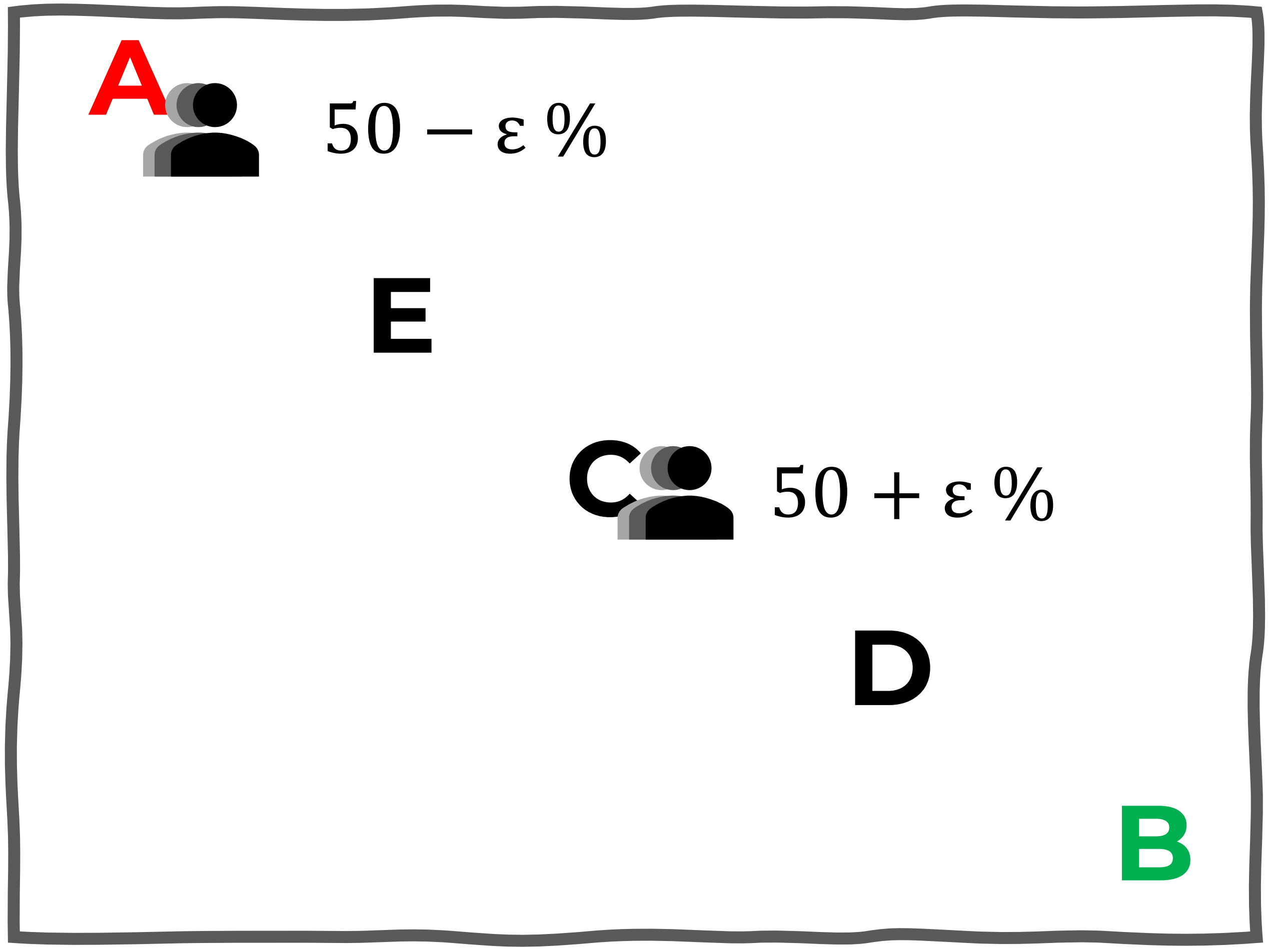
Worst-case Pairwise Distortion

Distortion of A and B

$50 - \varepsilon$ A ≻ E ≻ C ≻ D ≻ B

$50 + \varepsilon$ C ≻ D ≻ E ≻ B ≻ A

Question 1 (Cooperative)
Can we **decrease** this bound by choosing the positions of the candidates?



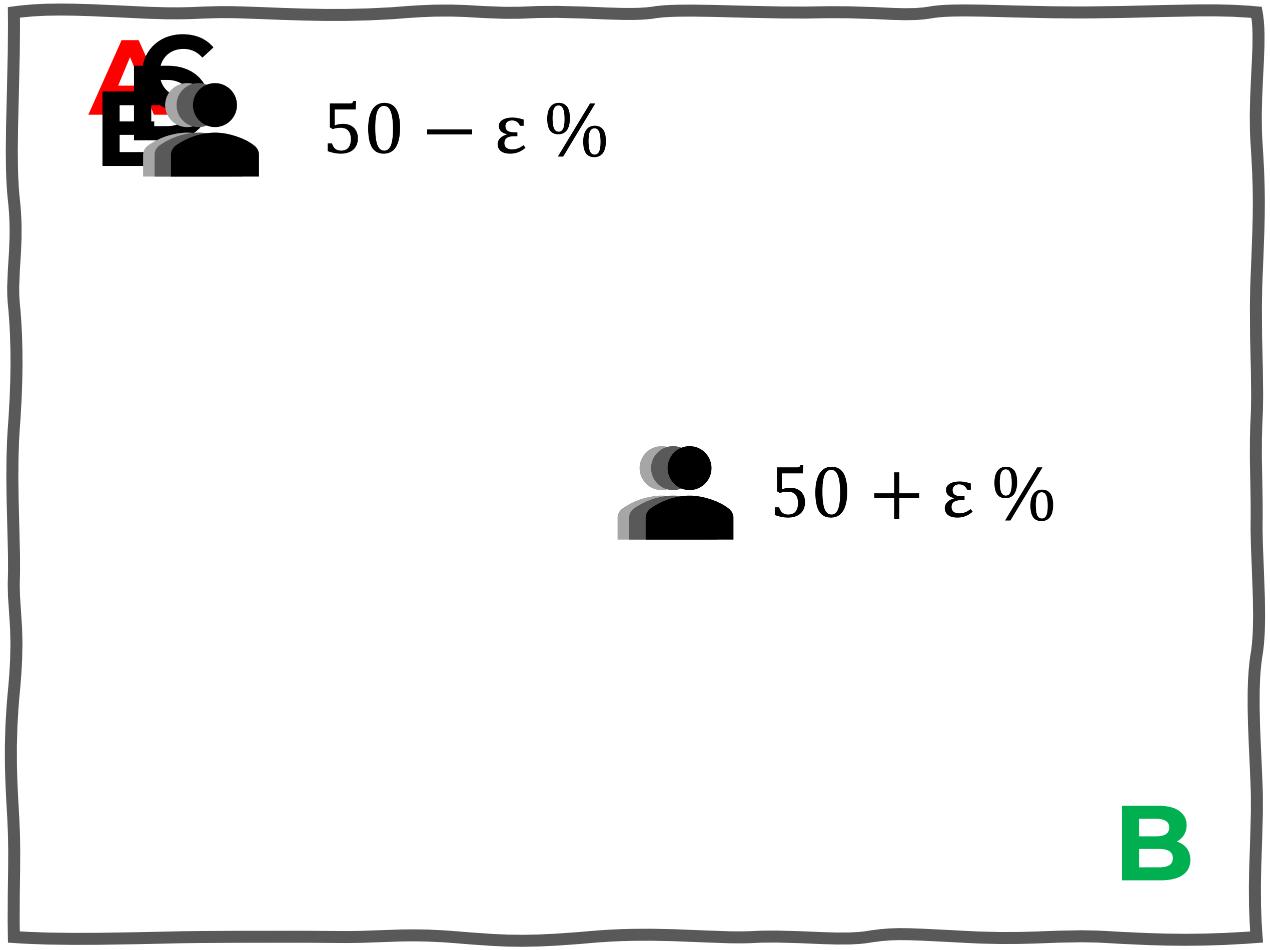
Worst-case Pairwise Distortion

Distortion of A and B

$50-\varepsilon$ C ≻ D ≻ E ≻ A ≻ B

$50+\varepsilon$ C ≻ D ≻ E ≻ B ≻ A

Question 2 (Adversarial)
Can we **make the bound worse** by choosing the positions of the candidates?

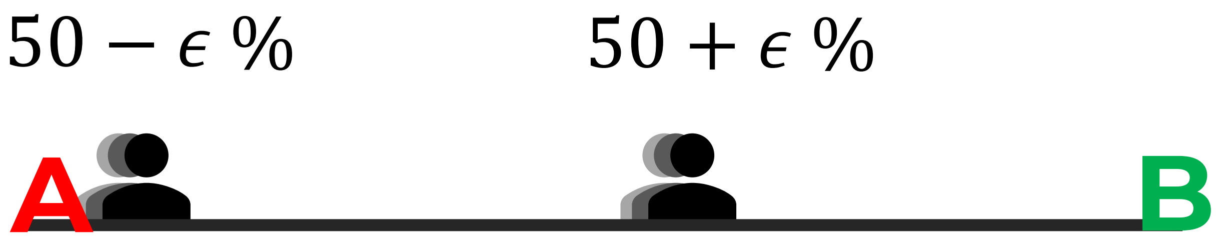


Worst-case Pairwise Distortion

We focus on the case of the **1D metric case** (the line)

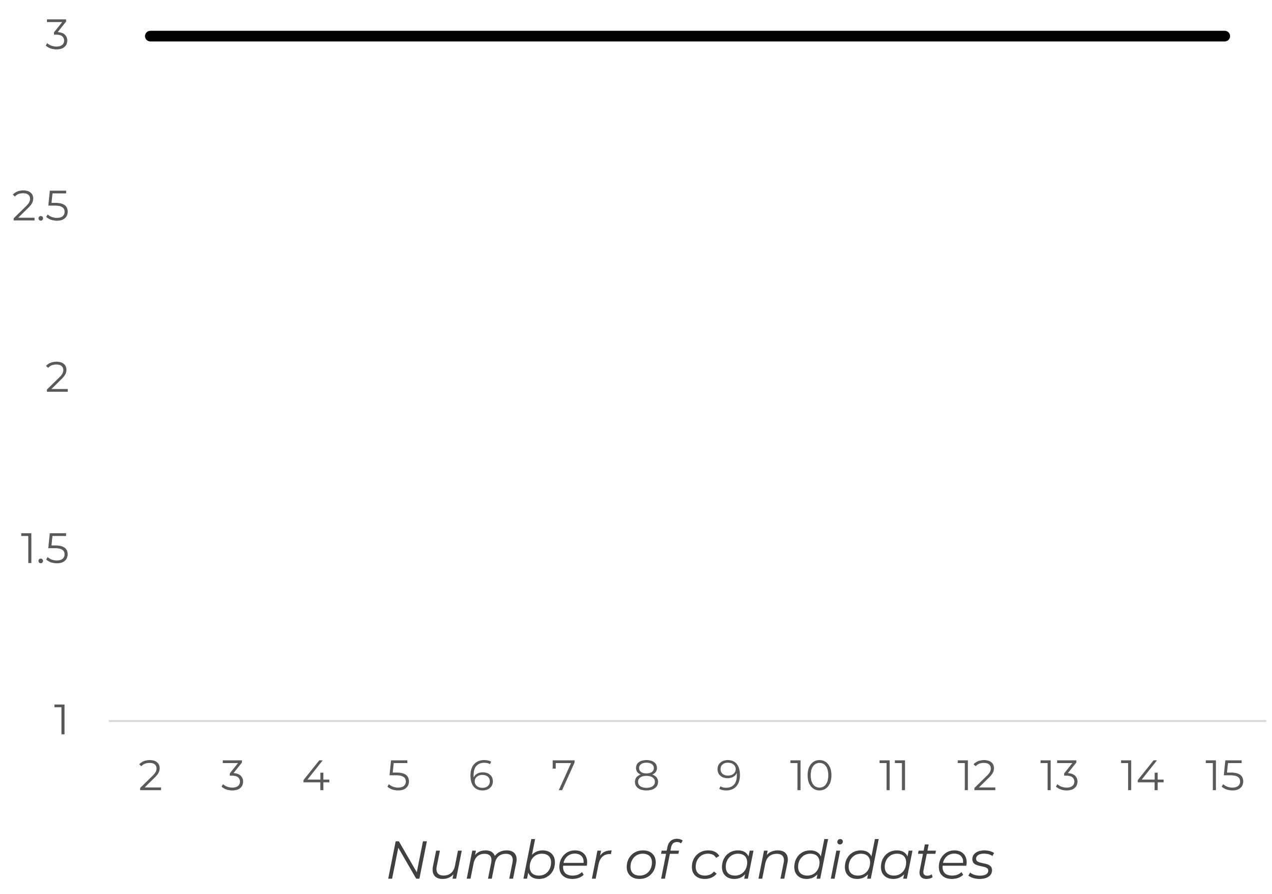
1. What is the worst-case distortion if a **cooperative** agent choose the positions of candidates (**optimistic** case)?

2. What is the worst-case distortion if an **adversarial** agent choose the positions of candidates (**pessimistic** case)?



Optimistic Positions of Candidates

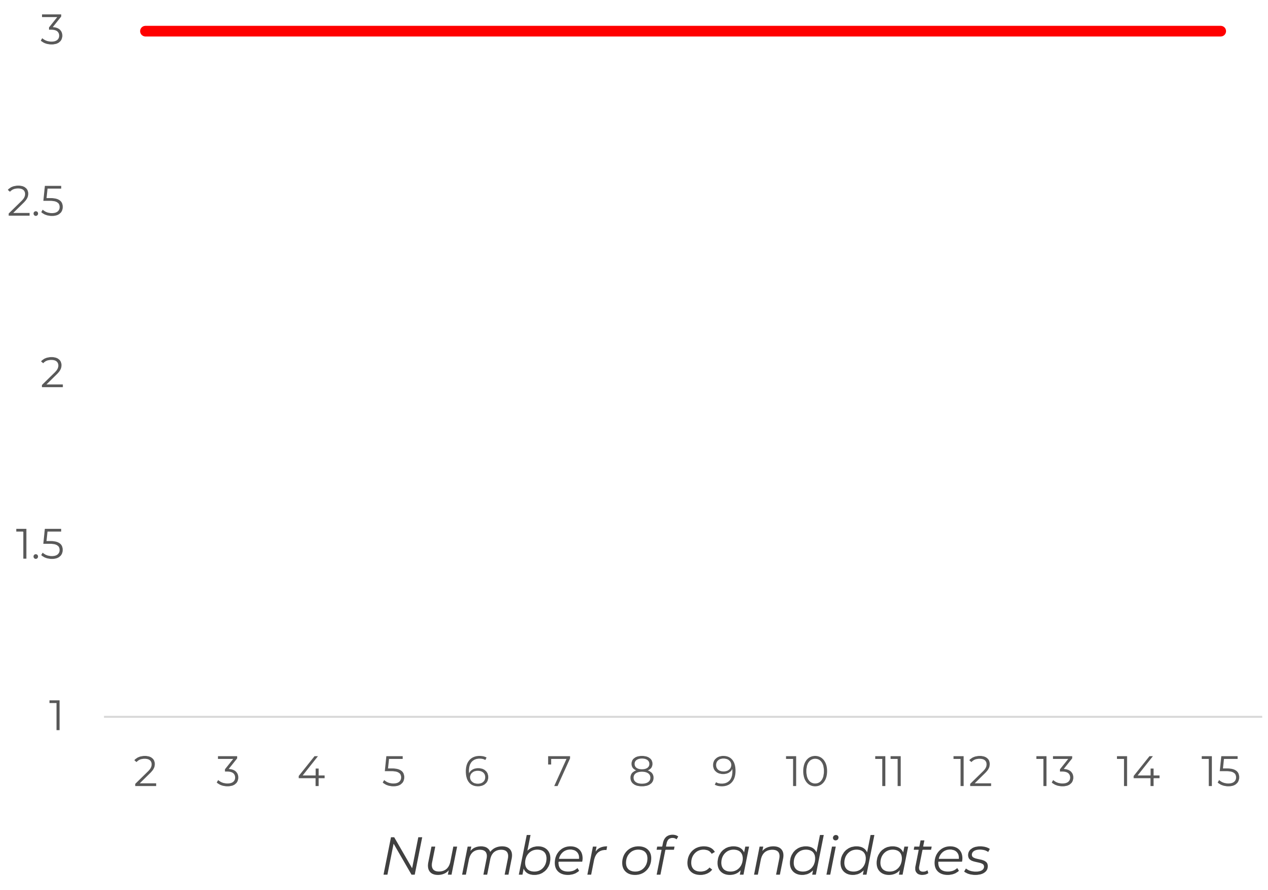
The **Majority Pairwise Rule** always has worst-case distortion of **3.**



Optimistic Positions of Candidates

The **Majority Pairwise Rule** always has worst-case distortion of **3.**

Plurality, **Veto** and **IRV** have the same distortion of **3** even with more candidates.



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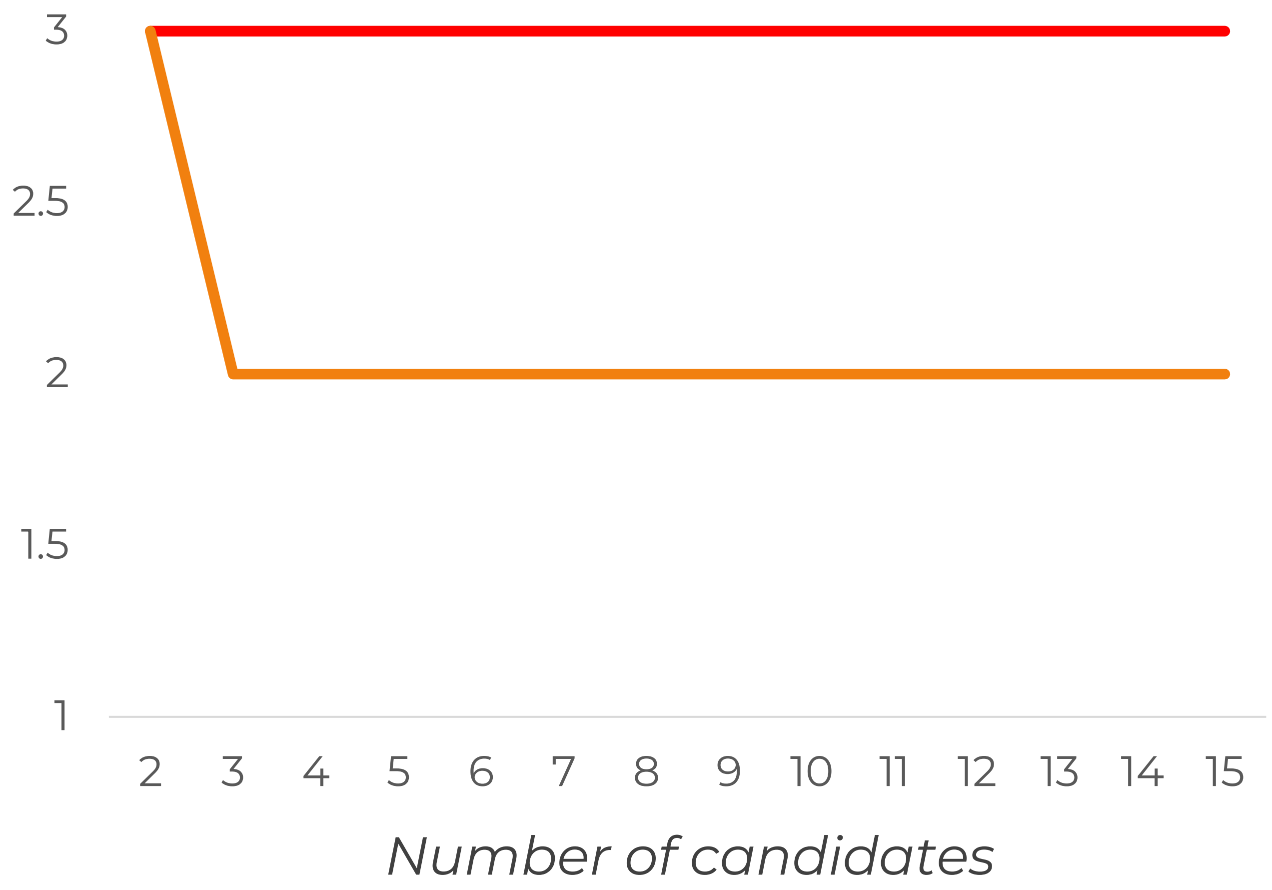
***k*-approval** has worst-case distortion **2** when $m \geq 3$.



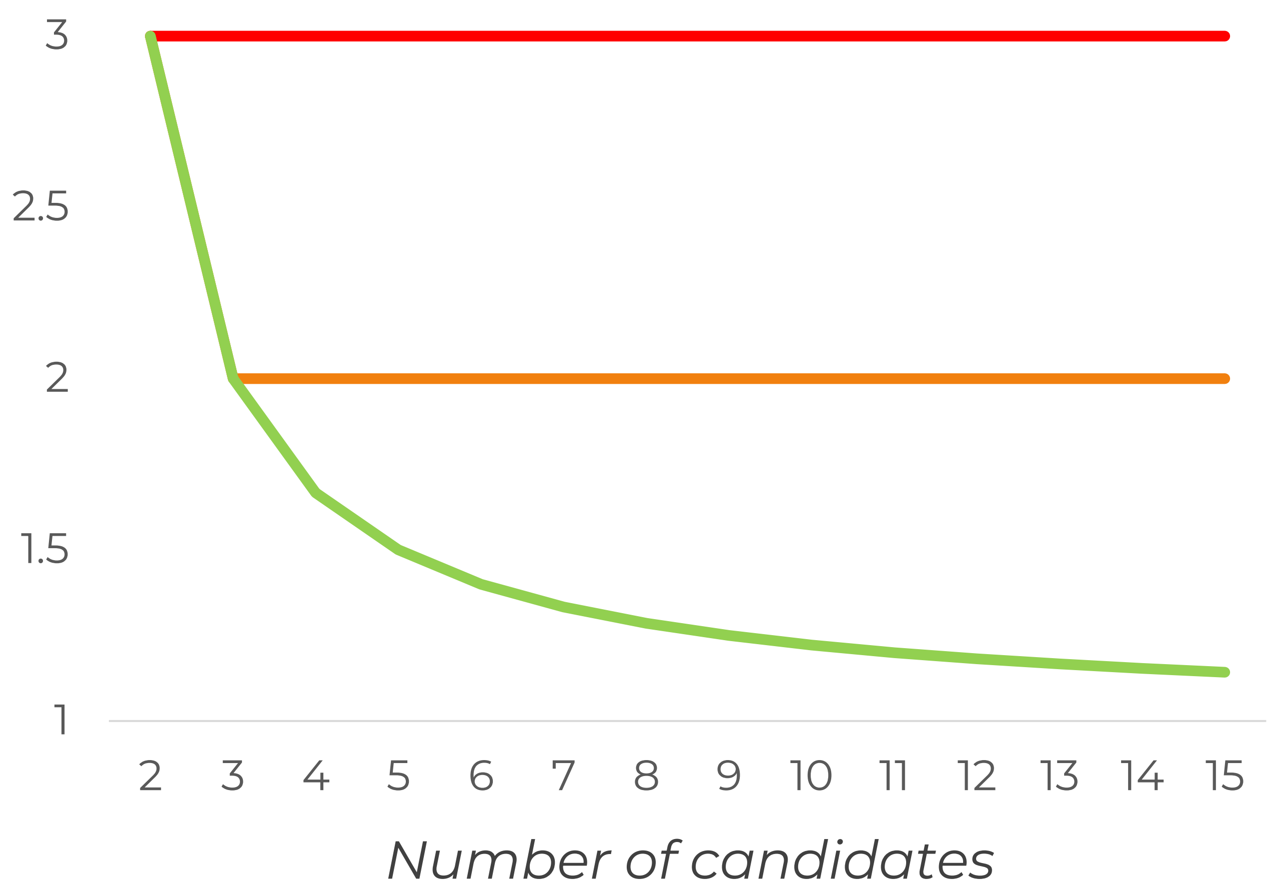
Optimistic Positions of Candidates

The **Majority Pairwise Rule** always has worst-case distortion of **3.**

Plurality, **Veto** and **IRV** have the same distortion of **3** even with more candidates.

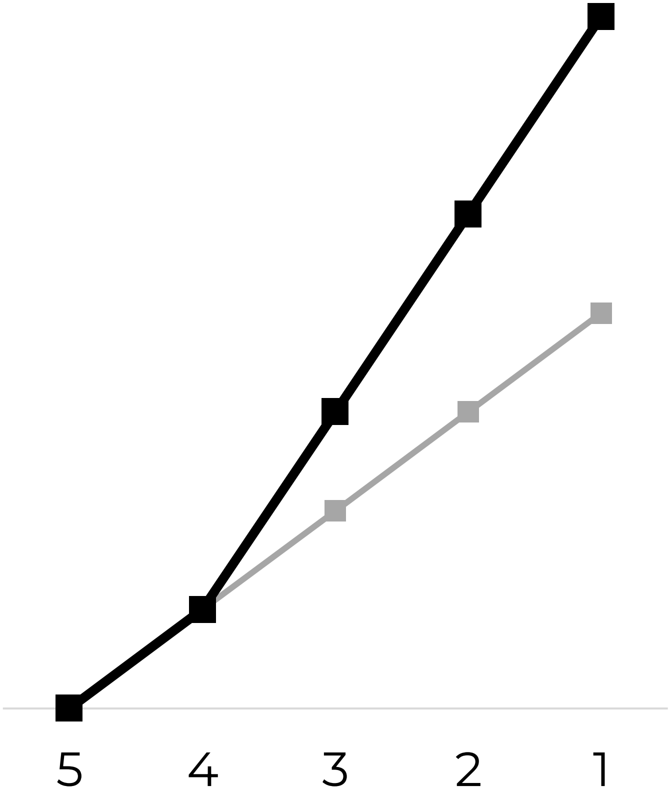
***k*-approval** has worst-case distortion **2** when $m \geq 3$.

Borda has worst-case distortion $\frac{m+1}{m-1}$.



Positional Scoring Rule

	A $\succ$	B $\succ$	C $\succ$	D $\succ$	E
Plurality	1	0	0	0	0
Veto	1	1	1	1	0
3-approval	1	1	1	0	0
Borda	4	3	2	1	0
Odd Borda	7	5	3	1	0

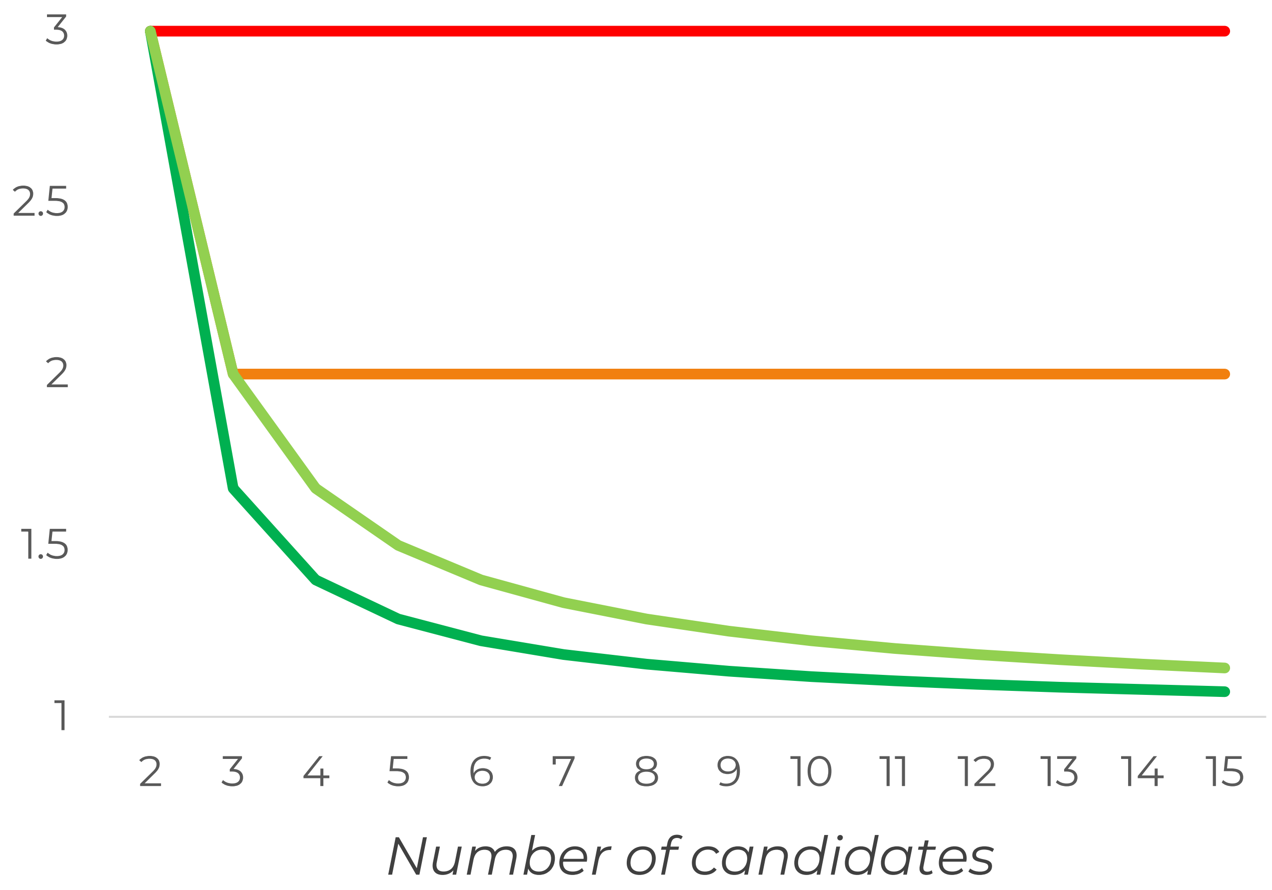


Optimistic Positions of Candidates

Odd-Borda is the optimal positional scoring rule with distortion $\frac{2m-1}{2m-3}$.

Conjecture

Odd-Borda is the optimal rule for $m \leq 5$.



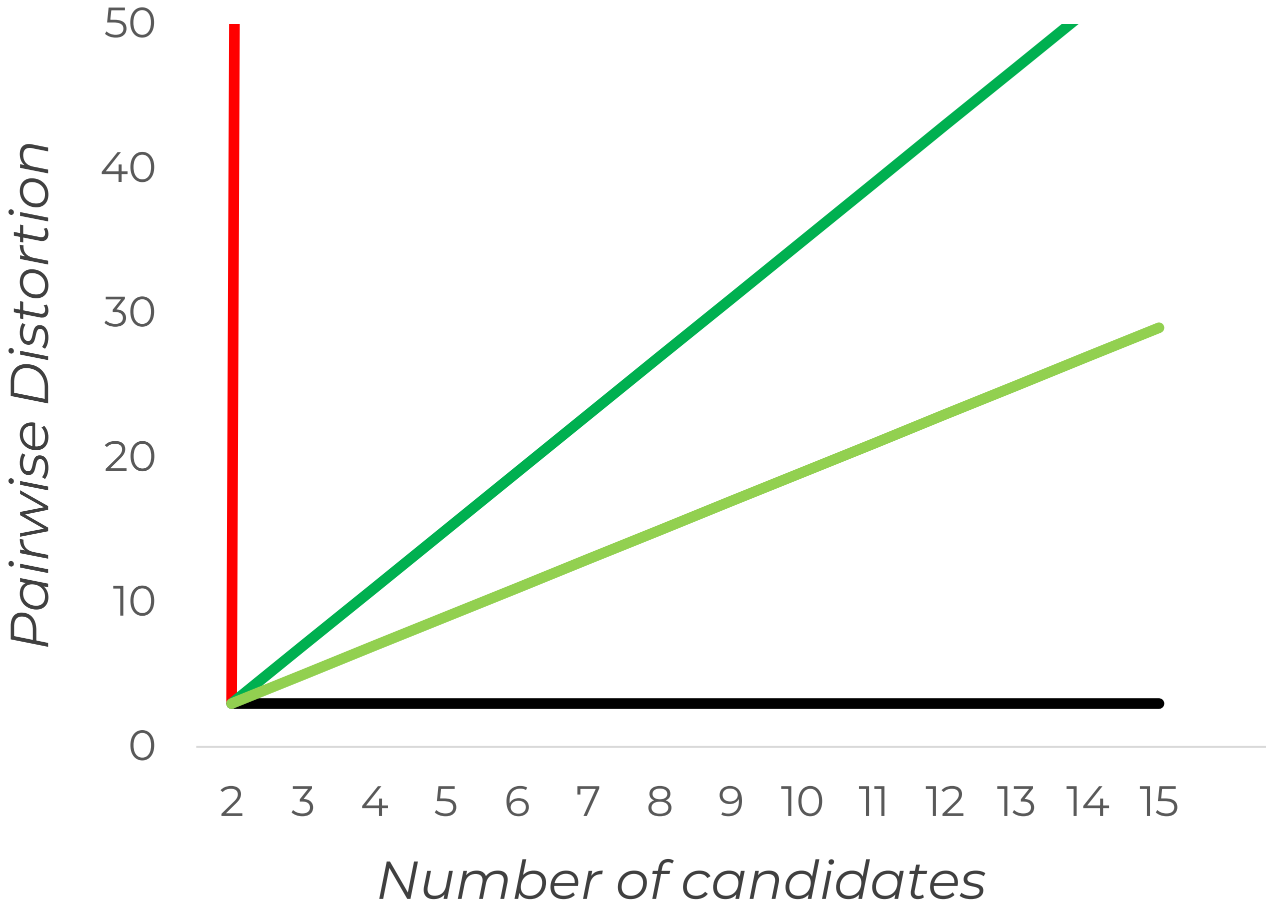
Pessimistic Positions of Candidates

The **Majority Pairwise Rule** always has worst-case distortion of **3.**

Plurality, **Veto**, **IRV** and ***k*-approval** have infinite distortion when $m \geq 3$.

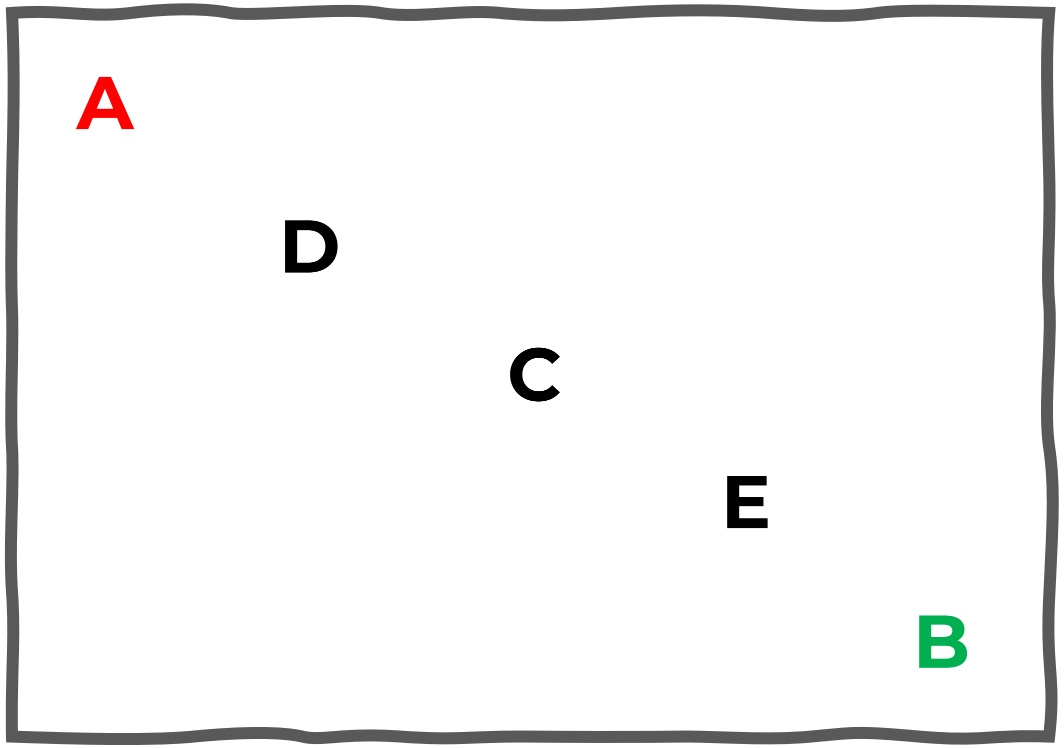
Borda has worst-case distortion $\mathbf{2m-1}$.

Odd Borda has worst-case distortion $\mathbf{4m-5}$.



Positions of the Candidates

Optimistic positions of candidates



Pessimistic positions of candidates

Conclusion

The impact of satisfying the IIA property on social welfare can be mild to very bad depending on the context.

Additional “irrelevant” candidates provide useful information, that is better used by some voting rules than others, such as **Borda**.

Is the IIA property only bad because it implies dictatorship, or is it a bit bad intrinsically (for social welfare)?

Questions?

(More cool things in the full paper available on my website)