Measuring a Priori Voting Power

Taking Delegations Seriously

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A voting game

Board of directors, each member has a voting weight:

	Ann	Bobby	Carol	Dan	Eve	Finn	Total
Weight	4	1	4	4	2	2	17
	+	+	-	+	+	+	13

Rule: a vote is successful if the sum of weights of voters in favor (+) is $\geq q = 12$.

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	+	+	-	+	+	+	13
	+	-	+	-	+	-	10

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	+	-	+	-	+	-	10

Rule: a vote is successful if the sum of weights of voters in favor (+) is $\geq q = 12$.

Bobby **never has any power** over the outcome of the vote.

A voting game

European Council of Ministers (1958), each member has a voting weight:

	France	Luxembourg	Germany	Italy	Belgium	Netherlands
Weight	4	1	4	4	2	2

Rule: a vote is successful if the sum of weights of voters in favor (+) is $\geq q = 12$.

Luxembourg **never has any power** over the outcome of the vote.

How to measure the voting power?

- *P-power*: [Shapley and Shubik, 1954].
- *I-power*: [Penrose, 1946, Banzhaf III, 1964, Coleman, 1971]

How to measure the voting power?

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More formally

Binary partition

A **binary partition** B is a map on V (voters) s.t. $B(i) \in \{-1, +1\}$ for all $i \in V$.

	Ann	Bobby	Carol	Dan	Eve	Finn
B	+	-	+	-	+	+

$B^- = \{\text{Bobby}, \text{Dan}\}$ and $B^+ = \{\text{Ann}, \text{Carol}, \text{Eve}, \text{Finn}\}$

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Binary voting rule

A **binary voting rule** W associates to every binary partition B an outcome $W(B) \in \{-1, +1\}$.

	Ann	Bobby	Carol	Dan	Eve	Finn	$W(B)$
B	+	-	+	-	+	+	$+1$

More formally

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B	+	-	+	-	+	+	$+1$
$B' \geq B$	+	+	+	-	+	+	$+1$

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B	+	-	+	-	+	+	$+1$
$B' \geq B$	+	+	+	-	+	+	$+1$
B_-	-	-	-	-	-	-	-1
B_+	+	+	+	+	+	+	$+1$

More formally

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Weighted voting rule (= Weighted Voting Game)

A **weighted voting rule** with weights $w : V \to \mathbb{N}$ and a quota $q \in \mathbb{N}$ is such that $W(B) = +1$ if and only if $\sum_{i \in B^+} w(i) \geq q$.

	Ann	Bobby	Carol	Dan	Eve	Finn	Total	$W(B)$
Weights	4	1	4	4	2	2	17	
B	+	-	+	-	+	+	12	$+1$

The **Penrose-Banzhaf measure** is the probability of a voter being able to alter the election's outcome given the following probabilistic model: *all binary partitions are equally likely to occur*.

Penrose-Banzhaf measure

Given a binary voting rule W, the **Penrose-Banzhaf measure** of voter $i \in V$ is defined as:

$$\mathcal{M}_i(W) = \sum_{B \in \mathcal{B}} \mathbb{P}(B) \frac{W(B_{i+}) - W(B_{i-})}{2},$$

where $\mathbb{P}(B) = 1/2^n$ for all partitions B.

	Ann	Bobby	Carol	Dan	Eve	Finn	$W(B)$
B	+	-	-	?	+	+	

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B	+	-	-	?	+	+	
B_{D-}	+	-	-	-	+	+	-1

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B	+	-	-	?	+	+	
B_{D-}	+	-	-	-	+	+	-1
B_{D+}	+	-	-	+	+	+	$+1$

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where $\mathbb{P}(B) = 1/2^n$ for all partitions B.

Complexity

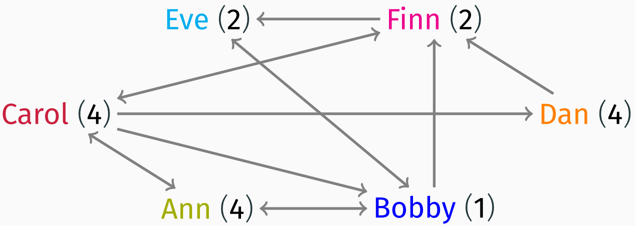
- #P-hard in general [Prasad and Kelly, 1990],
- In **WVGs**, it can be computed by a pseudo-polynomial algorithm that runs in polynomial time w.r.t. $|V|$ and $\max_{i \in V} w(i)$ [Matsui and Matsui, 2000].

A voting game

	Ann	Bobby	Carol	Dan	Eve	Finn
Weight	**4**	**1**	**4**	**4**	**2**	**2**
$\mathcal{M}_i(W)$	0.3125	0	0.3125	0.3125	0.1875	0.1875

Rule: a vote is successful if the sum of weights of voters in favor (+) is $\geq q = 12$.

Liquid Democracy



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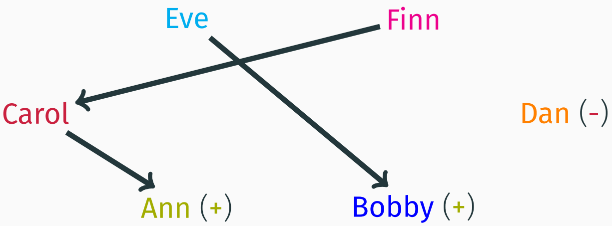

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Weight	4	1	4	4	2	2	17
Ballots	+	+	A	-	B	C	

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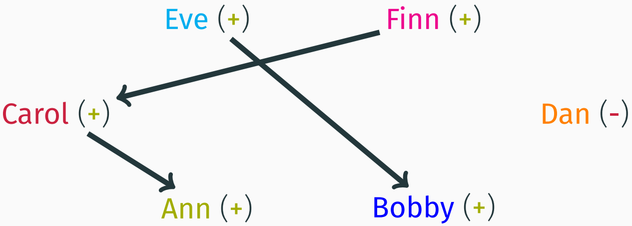

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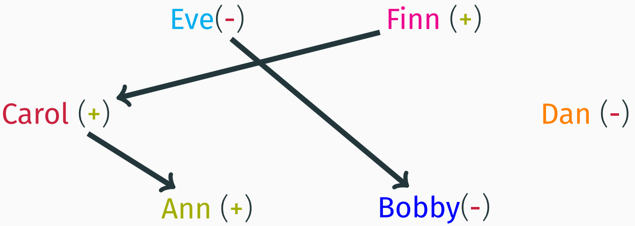

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Ballots	+	-	A	-	B	C	
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Motivation behind delegations

Voting models using **delegations** are getting increasing attention, both in theoretical works and in practice:

- In **Proxy Voting (PV)**, there is a fixed set of representatives to whom voters can delegate their votes.
- In **Liquid Democracy (LD)**, every voter can either vote directly or delegate its voting power to someone else.

Formal definition

We assume that we have a graph structure $G = (V, E)$ in which each voter $v \in V$ can vote **for**, **against** or **delegate** to a neighbour.

G-delegation partition

A G-**delegation partition** D is a map on V (voters) s.t. $D(i) \in \{-1, +1\} \cup \mathrm{NB}_{out}(i)$.

$\mathrm{NB}_{out}(i)$: set of out-neighbours of $i \in V$.

	Ann	Bobby	Carol	Dan	Eve	Finn
D	+	+	A	-	B	C

$D^{-} = \{\text{Dan}\}$, $D^{+} = \{\text{Ann}, \text{Carol}\}$,

$D^{\text{Bobby}} = \{\text{Eve}\}$, $D^{\text{Ann}} = \{\text{Carol}\}$ and $D^{\text{Carol}} = \{\text{Finn}\}$

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Direct vote partition

A **direct vote partition** T is a map on V s.t. $T(i) \in \{-1, 0, +1\}$.

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D	+	+	A	-	B	C
T_D	+	+	+	-	+	+

$\Rightarrow$ A G-delegation partition D naturally induces a direct-vote partition T_D.

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	Ann	Bobby	Carol	Dan	Eve	Finn
D	+	+	F	-	B	C
T_D	+	+	0	-	+	0

$\Rightarrow$ A G-delegation partition D naturally induces a direct-vote partition T_D.

Liquid Democracy (LD) Penrose-Banzhaf measure

Given a digraph $G = (V, E)$ and a ternary voting rule W, the **LD Penrose-Banzhaf measure** of voter $i \in V$ is defined as:

$$\mathcal{M}_i^{ld}(W, G) = \sum_{D \in \mathcal{D}} \mathbb{P}(D) \frac{W(T_{D_{i+}}) - W(T_{D_{i-}})}{2},$$

where $\mathbb{P}(D)$ is the probability of the G-delegation partition D occurring.

- Probability to **delegate** $p_d^i \in [0, 1]$ and to **vote** $p_v^i = 1 - p_d^i$.
- If **vote**: probability to vote **for**/**against**: $p_+ = p_- = 1/2$.
- If **delegate**: probability to delegate to $j \in \text{NB}_{out}(i)$: $1/|\text{NB}_{out}(i)|$.

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- If **delegate**: probability to delegate to $j \in \mathrm{NB}_{out}(i)$: $1/|\mathrm{NB}_{out}(i)|$.

If $p_d^i = 0$ for every voter $i \in N$, we have the classic Penrose-Banzhaf index.

Complexity

Computing the LD Penrose-Banzhaf:

- **#P-hard**, even for Weighted Voting Games (WVG).

Complexity and computation

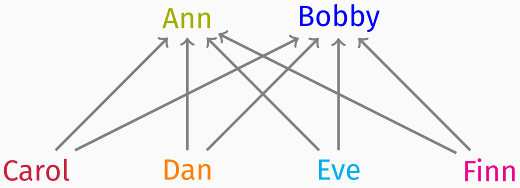
Complexity

Computing the LD Penrose-Banzhaf:

- #P-hard, even for Weighted Voting Games (WVG).
- For bipartite and complete graphs, it can be computed by a pseudo-polynomial algorithm that runs in polynomial time w.r.t. $|V|$ and $\max_{i \in V} w(i)$.

Proxy Voting

In **Proxy Voting (PV)**, we have *delegatees* $i \in V_d$ (*proxies*) and *delegators* $i \in V_v$.



Complexity

The LD Penrose-Banzhaf can be computed by a pseudo-polynomial algorithm that runs in polynomial time w.r.t. $|V|$ and $\max_{i \in V} w(i)$.

Proxy Voting: Experiments

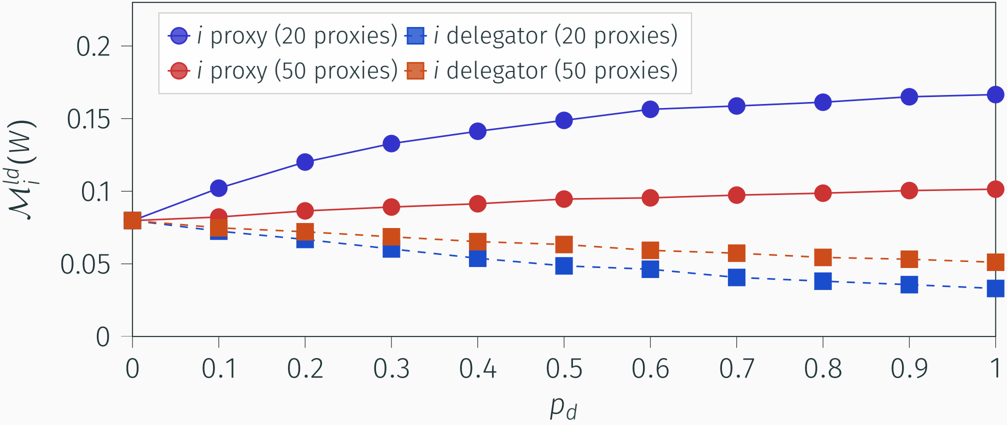
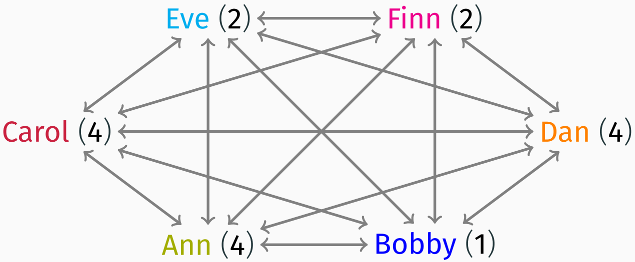


Figure 1: 100 voters, WVG with all weights equal to 1 and $q = 50\%$.

The lower the number of proxies,
the **more unequal the voting power of the voters**.

Liquid Democracy

In **Liquid Democracy**, any voter can delegate to any other voter, or vote themselves.



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Liquid Democracy: experiments

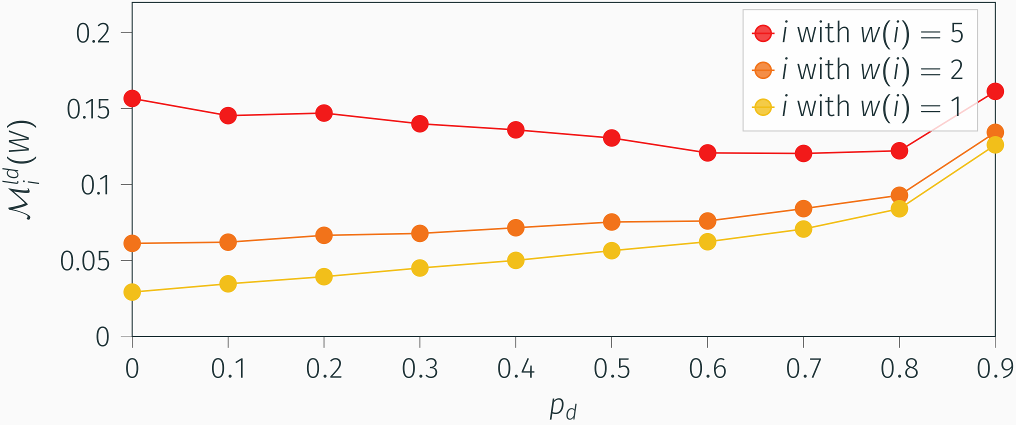


Figure 2: Penrose-Banzhaf index of the voters with probability to delegate p_d. 100 voters, WVG with 50 (resp. 30, 20) voters with weights equal to 1 (resp. 2, 5) and $q = 50\%$.

When the probability to delegate p_d gets higher,
the **voting weight** has less influence on the voting power.

Criticality distribution and degree distribution

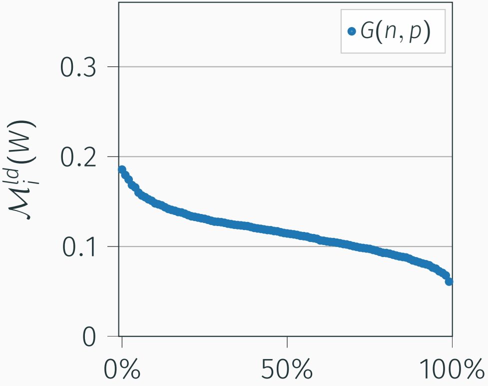
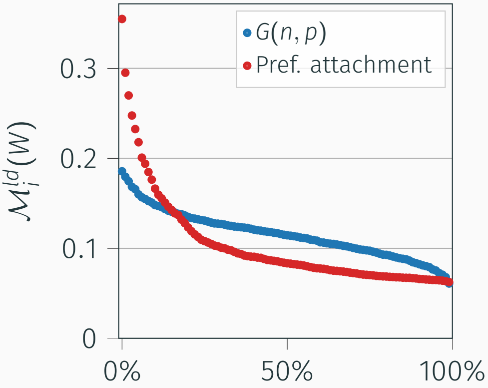


Figure 3: Distribution of the criticality of the voters in the network, from the highest degree to the smallest criticality

Random graph $G(n,p)$

- Undirected.
- Every edge has probability p to exist.

Criticality distribution and degree distribution



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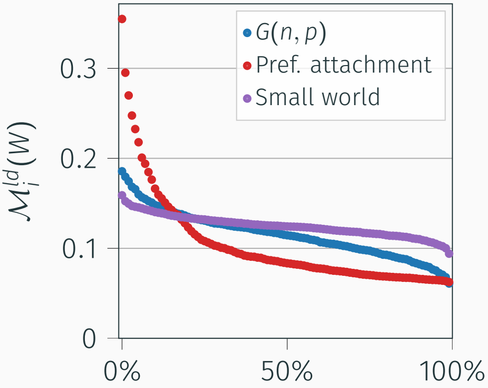
Preferential attachment model

- [Barabási and Albert, 1999].
- Undirected.
- Voters join the network one by one and are more likely to be linked to already popular voters.
- *"Rich get richer"*

Criticality distribution and degree distribution



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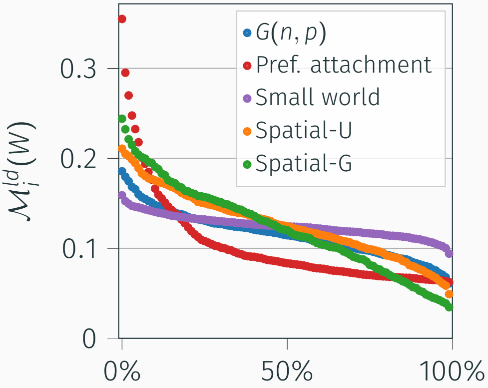
Small world model

- [Watts and Strogatz, 1998] .
- Undirected.
- Voters on a circle and linked in priority to their neighbours on the circle.

Criticality distribution and degree distribution



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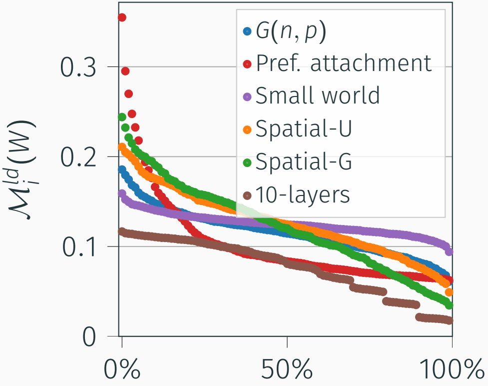
Spatial models

- Directed.
- Voters randomly placed on a 2D-plane (*Uniform* or *Gaussian* distribution).
- Voters have a directed edge towards their k nearest neighbours.

Criticality distribution and degree distribution



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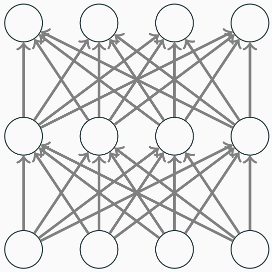
k-layers models



example with $k = 3$ layers.

Criticality distribution and degree distribution

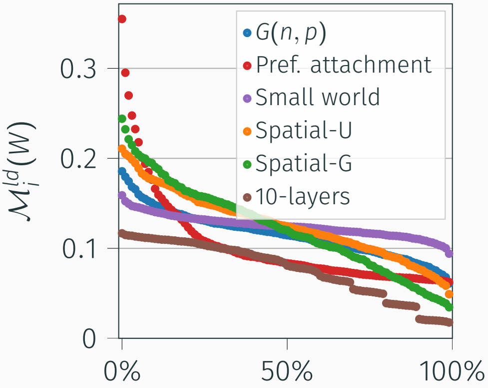


Figure 3: Distribution of the criticality of the voters in the network, from the highest criticality to the smallest criticality

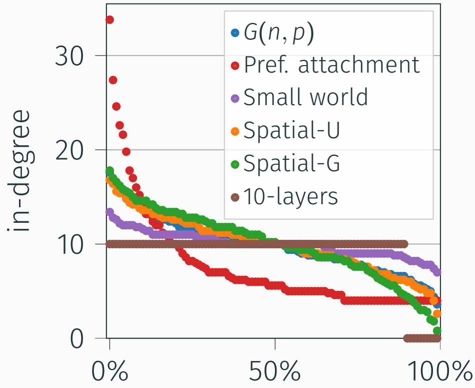


Figure 4: Distribution of the degree of the voters in the network, from the highest degree to the smallest degree

This paper continues the tradition of extending the notion of a priori voting power to new voting models.

- We have introduced the *Liquid Democracy Penrose-Banzhaf measure* to evaluate **how critical voters are** in deciding the outcome of an election where delegations play a key role.
- **Complexity** and hardness results, and pseudo-polynomial algorithms for PV and LD.
- **Experimental analysis** of the criticality in various networks and with varying parameters.

Conclusion

Further research directions:

- Other delegations **models** (ranked delegations, including abstention, etc.).
- Finding conditions (like adding or removing neighbours) that **affects the power measure**.
- Analysing **real data**, using real networks for instance.

Thanks for your attention!

Questions?

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