

# Notions of Single-Peakedness for Incomplete Preferences

Théo Delemazure<sup>1</sup> and Ulle Endriss<sup>1</sup> and Umberto Grandi<sup>2</sup> and Mohamed Ouaguenouni<sup>2</sup>

<sup>1</sup>ILLC, University of Amsterdam

<sup>2</sup>IRIT, Université de Toulouse

{theo.delemazure, ulle.endriss}@uva.nl, {umberto.grandi, mohamed-elhachemi.ouaguenouni}@irit.fr

## Abstract

Single-peakedness is one of the core concepts studied in social choice theory. When the preferences of a population are single-peaked, this greatly simplifies the process of collective decision making and can offer deep insights into the internal makeup of that population. Inspired by the needs of applications in the domain of digital democracy, we investigate how to generalise the concept of single-peakedness from the classical model of voting with ranked preferences to a richer and more flexible model where preferences can be incomplete and where they can include indifferences. We show that a systematic analysis leads to a total of ten different notions of single-peakedness, which we study in axiomatic, algorithmic, and experimental terms.

## 1 Introduction

Social choice theory provides us with a rich set of tools to analyse and aggregate the preferences of groups of individuals [Arrow *et al.*, 2002; Brandt *et al.*, 2016]. But the model of preference typically assumed, where each individual ranks all of the available alternatives from best to worst, is an idealisation of the preferences we encounter in the real world. In practice, a voter might be indifferent between two alternatives or they might not rank them at all. This is true, especially, for the preferences we might collect using an online platform, where users spend little time while being confronted with many alternatives. It is important to develop the theory of social choice also for such richer models of preference—not least in view of ongoing efforts at the interface of AI with Political Science to develop innovative solutions enabling truly participatory democracy [Grossi *et al.*, 2024].

We study the concept of single-peakedness [Gaertner, 2001; Elkind *et al.*, 2025]. A profile of preferences, one for each voter, is single-peaked if we can arrange the alternatives on which voters express preferences along an axis in such a way that, for each voter, the alternatives to the left of their most preferred alternative are arranged in increasing order and those to its right in decreasing order of preference. Single-peakedness is important for two reasons. First, when preferences are single-peaked, we can circumvent many of

the field’s negative results, such as Arrow’s Theorem, meaning it becomes easier to design good rules for collective decision making. Second, being able to say along which axis a given preference profile is single-peaked tells us much about the alternatives and the internal make-up of the group of voters ranking them. So single-peakedness is useful both for decision making and for preference analysis.

Our purpose here is to explore possible definitions of single-peakedness when preferences might be incomplete and when they might allow for indifferences. As we shall see, there is no single obviously correct definition. We map out the space of possible definitions and study their relationships, we analyse the extent to which they can offer similarly positive guarantees as the definition for the classical model of total preferences, and we use them to analyse real-world preference data, where the availability of multiple definitions allows us to pinpoint different phenomena of interest.

We are going to identify ten different definitions of single-peakedness. Why so many? In a nutshell, the idea is the following (for full details see Section 3). In the classical model, a preference profile is single-peaked if we can arrange the alternatives along some axis such that, whenever alternative  $b$  is *between* alternatives  $a$  and  $c$  on that axis, every voter will rank  $b$  strictly above  $a$  or strictly above  $c$  (or both):  $(b \succ a) \vee (b \succ c)$ . In the classical model, whether we ask for a strict preference of  $b$  over  $a$ , or a weak one—or indeed the absence of either a strict or a weak preference of  $a$  over  $b$ —makes no difference. But for our richer model, where we can distinguish between strict and weak preferences, and between present and absent preferences, any combination of choices we make might, in principle, lead to a new notion of single-peakedness. A careful analysis, avoiding double-counting of equivalent definitions obtained by alternative means, yields a total of ten notions of single-peakedness. If we restrict attention to weak preferences (no incomparabilities), this collapses to three notions, and the same is true if we restrict attention to partial preferences (no indifferences).

We demonstrate that this systematic enumeration leads to meaningful notions of single-peakedness of varying strength. We also show that the computational difficulty of recognising whether a given profile of preferences is single-peaked (relative to some suitable axis) differs for different notions of single-peakedness. Maybe the main reason for the popularity of single-peakedness in the literature is the fact that, in

the classical model, single-peakedness is a sufficient condition for avoiding the Condorcet Paradox: it ensures that we can always select an alternative  $x$  such that no other alternative would beat  $x$  in a direct majority contest. At a technical level, this is due to single-peakedness guaranteeing the transitivity of the majority relation, and we analyse for which of our single-peakedness conditions we can (or cannot) give corresponding guarantees. Finally, we use our definitions to analyse real-world preference data in two case studies. Our analysis shows that the choice of single-peakedness notion can heavily impact results, in one case suggesting a possible re-interpretation of the findings of an important study on consensus finding in deliberative democracy [List *et al.*, 2013].

**Related work.** The notion of single-peakedness was introduced by Black [1948], who argued for it to be seen as both a natural and a realistic domain restriction, and who showed that it circumvents the Condorcet Paradox. While nowadays it is well-understood that real-world electorates cannot be expected to have perfectly single-peaked preferences [Feld and Grofman, 1986]—an effect we are also going to observe in the data we analyse in this paper—the usefulness of single-peakedness as a tool for political analysis is nonetheless broadly accepted [Farrar *et al.*, 2010; Rad and Roy, 2021]. Indeed, single-peakedness is widely studied in computational social choice [Elkind *et al.*, 2025]. But a specific focus on incompleteness is rare when it comes to the study of domain restrictions—even if more common in other parts of the field [Lang, 2020; Terzopoulou, 2021].

Most closely related to our work, Fitzsimmons and Lackner [2020] study the complexity of recognising single-peaked profiles in a similar model as ours. But there are two important differences. First, they do not consider general preferences that allow for *both* indifference and incomparability. Second, they interpret the absence of a preference as epistemic incompleteness rather than inherent incompleteness, and accordingly focus on the notion of *possible* single-peakedness (looking for a completion of the partial-preference profile that is single-peaked in the classical sense).

Elkind and Lackner [2015] study domain restrictions, including single-peakedness, for *dichotomous* preferences, which are a special class of weak preferences. Their focus is on the impact such domain restrictions have on the complexity of outcome determination for several voting rules.

**Paper overview.** After recalling common notation and terminology in Section 2, we develop our definitions of single-peakedness and analyse their relative strengths in Section 3. Then, Section 4 is dedicated to our complexity results, Section 5 to the analysis of the transitivity of the simple and the absolute majority relations, and Section 6 to our case studies on real preference data. Section 7 concludes. Additional information is available in the supplementary material.

## 2 Preliminaries

In this section we define our model of voting, where preferences can be incomplete and where voters can declare indifference between pairs of alternatives. This generalises the classical model of voting with total orders [Zwicker, 2016].

Let  $N = \{1, \dots, n\}$  be a set of  $n$  voters and  $X$  a set of  $m$  alternatives. Each voter  $i \in N$  has a preference  $\succsim_i$ , which we take to be a *preorder* on  $X$ , i.e., a binary relation that is reflexive and transitive. Given  $\succsim_i$ , we write  $a \succ_i b$  (strict preference) in case  $a \succsim_i b$  but not  $b \succsim_i a$ . We often omit the index  $i$  from our notation in case the voter’s identity is clear from context or not relevant. If both  $a \succsim_i b$  and  $b \succsim_i a$  hold, we say that voter  $i$  is *indifferent* between  $a$  and  $b$ . If neither  $a \succsim_i b$  nor  $b \succsim_i a$  hold, we say that  $a$  and  $b$  are *incomparable* from the point of view of voter  $i$ .

In addition to being reflexive and transitive, a preference  $\succsim_i$  might also be *complete* ( $a \succsim_i b$  or  $b \succsim_i a$  for all  $a, b \in X$ ), which rules out the possibility of incomparability, and it might be *antisymmetric* (both  $a \succsim_i b$  and  $b \succsim_i a$  only if  $a = b$ ), which rules out the possibility of indifference. We say that  $\succsim_i$  is a *weak order* if it is complete, a *partial order* if it is antisymmetric, and a *total order* if it is both.

A *profile* is a vector  $\succsim = (\succsim_1, \dots, \succsim_n)$  of preferences, one for each voter. For a given profile  $\succsim$  and alternatives  $a, b \in X$ , we write  $N_{a \succ b}^\succsim$  for the subelectorate consisting of the set of voters who rank  $a$  above  $b$  in profile  $\succsim$ .

## 3 Notions of Single-Peakedness

In this section we derive multiple notions of single-peakedness for our enriched model as natural generalisations of the classical notion of single-peakedness for total orders.

An *axis* is a strict linear order  $\triangleright$  on the set of alternatives  $X$ . We identify axes that differ only by reversal (in this sense, the axes  $\triangleright_1 = abcd$  and  $\triangleright_2 = dcba$  are equivalent). Given an axis  $\triangleright$ , we say that alternative  $b \in X$  is *between*  $a$  and  $c$  if  $a \triangleright b \triangleright c$  or  $c \triangleright b \triangleright a$ , and we denote this fact by  $\text{Bet}^\triangleright(a, b, c)$ —or equivalently by  $\text{Bet}^\triangleright(c, b, a)$ .

Black’s original notion of single-peakedness of a profile of preferences [Black, 1948] was intended for preferences that are total orders. It relies on the idea that each voter has a most preferred alternative (the *peak*) and that their preference decreases as we move away from this peak, along either direction of the axis. As is well-known [Elkind *et al.*, 2025], this condition can equivalently be stated in terms of *betweenness*: if  $b$  is between  $a$  and  $c$  on the axis, then every voter must strictly prefer  $b$  to  $a$  or strictly prefer  $b$  to  $c$  (so there must be no ‘valley’ in their preference). In other words, a profile  $\succsim$  of total orders is single-peaked if  $\text{Bet}^\triangleright(a, b, c)$  implies  $(b \succ_i a) \vee (b \succ_i c)$  for all voters  $i \in N$ .

### 3.1 Generalising from Total Orders to Preorders

When generalising from total orders to arbitrary preorders, conditions such as  $(b \succ_i a)$  can be rephrased in different ways that would all be equivalent for the special case of total orders. For instance, one might wish to accept that voter  $i$  is indifferent between  $a$  and  $b$ , or that they consider  $a$  and  $b$  incomparable. For a preference  $\succsim$  and an ordered pair of alternatives  $(x, y) \in X \times X$ , we refer to the four possible forms of preference using these predicates:

$$\begin{aligned} \mathbf{S}(x, y) &\Leftrightarrow (x \succ y) && \text{(strict preference in favour)} \\ \mathbf{W}(x, y) &\Leftrightarrow (x \succsim y) && \text{(weak preference in favour)} \\ \mathbf{\bar{W}}(x, y) &\Leftrightarrow \neg(y \succ x) && \text{(no weak preference against)} \\ \mathbf{\bar{S}}(x, y) &\Leftrightarrow \neg(y \succ x) && \text{(no strict preference against)} \end{aligned}$$

When clear from context, we usually omit  $x$  and  $y$  and write  $S$  rather than  $S(x, y)$ , and so forth. Note that these predicates satisfy the implications  $S \Rightarrow W \Rightarrow \bar{S}$  and  $S \Rightarrow \bar{W} \Rightarrow \bar{S}$ , while  $W$  and  $\bar{W}$  are incomparable to one another. Also note that for total orders (and  $x \neq y$ ) we get  $S = W = \bar{W} = \bar{S}$ .

These four different ways of generalising from the simple form of preference appropriate when talking only about total orders allow us to generalise the classical definition of single-peakedness to preorders in a systematic way. For any two predicates  $X, Y \in \{S, W, \bar{W}, \bar{S}\}$  and any axis  $\triangleright$  on  $X$ , we say that preference  $\succsim$  is  $XY$ -single-peaked if it satisfies the following condition for any distinct alternatives  $a, b, c \in X$ :

$$\text{Bet}^\triangleright(a, b, c) \implies X(b, a) \vee Y(b, c)$$

Thus, if  $b$  is between  $a$  and  $c$  (for either direction), we must respect at least one of the two preference statements for  $b$  over the other alternatives. A profile  $\succsim = (\succsim_1, \dots, \succsim_n)$  is  $XY$ -single-peaked if  $\succsim_i$  is  $XY$ -single-peaked for every  $i \in N$ .

This definition yields a total of  $4 \times 4 = 16$  notions of single-peakedness. However, for any two  $X, Y \in \{S, W, \bar{W}, \bar{S}\}$ ,  $XY$ -single-peakedness is logically equivalent to  $YX$ -single-peakedness, by symmetry of betweenness. Thus, we can reduce the number of distinct notions to (at most) ten. (We will soon see that it is *exactly* ten.)

The implications  $S \Rightarrow W \Rightarrow \bar{S}$  and  $S \Rightarrow \bar{W} \Rightarrow \bar{S}$  induce the lattice on the remaining single-peakedness conditions shown in Figure 1: if  $X \Rightarrow X'$  and  $Y \Rightarrow Y'$ , then  $XY$ -single-peakedness implies  $X'Y'$ -single-peakedness. But we get one additional implication,  $WW \Rightarrow \bar{S}\bar{S}$ , which is due to the symmetry of the betweenness-relation. Indeed, assume  $WW$ -single-peakedness holds and let  $b$  be between  $a$  and  $c$ . Due to  $\text{Bet}^\triangleright(a, b, c) \Leftrightarrow \text{Bet}^\triangleright(c, b, a)$ , condition  $WW$  entails:

$$\begin{aligned} & (W(b, a) \vee \bar{W}(b, c)) \wedge (W(b, c) \vee \bar{W}(b, a)) \\ \Leftrightarrow & (W(b, a) \wedge W(b, c)) \vee (W(b, a) \wedge \bar{W}(b, a)) \vee \\ & (\bar{W}(b, c) \wedge W(b, c)) \vee (\bar{W}(b, c) \wedge \bar{W}(b, a)) \\ \Rightarrow & W(b, c) \vee S(b, a) \vee S(b, c) \vee \bar{W}(b, c) \\ \Rightarrow & \bar{S}(b, c) \vee S(b, a) \vee \bar{S}(b, c) \vee \bar{S}(b, c) \\ \Leftrightarrow & S(b, a) \vee \bar{S}(b, c) \end{aligned}$$

We thus obtain  $\bar{S}\bar{S}$ , meaning that  $WW \Rightarrow \bar{S}\bar{S}$  holds.

For the case of arbitrary preorders, we cannot reduce the number of single-peakedness conditions any further. Indeed, for each of the arrows in Figure 1 there exists an example for a preference that is single-peaked for one but not the other condition concerned (see supplementary material).

However, if we restrict attention to certain special classes of preorders, the ten conditions collapse along two axes: on weak orders,  $S \equiv \bar{W}$  and  $W \equiv \bar{S}$  (as there are no incomparabilities); on partial orders,  $S \equiv W$  and  $\bar{S} \equiv \bar{W}$  (as there are no indifferences to distinguish strict from weak preference). So for both weak and partial orders, the ten notions of single-peakedness reduce to three equivalence classes each (as also shown in Figure 1). Finally, as previously mentioned already, for total orders, all ten notions coincide.

In the supplementary material, we provide informal descriptions and illustrations of what preferences ‘look like’ under each of our ten notions of single-peakedness.

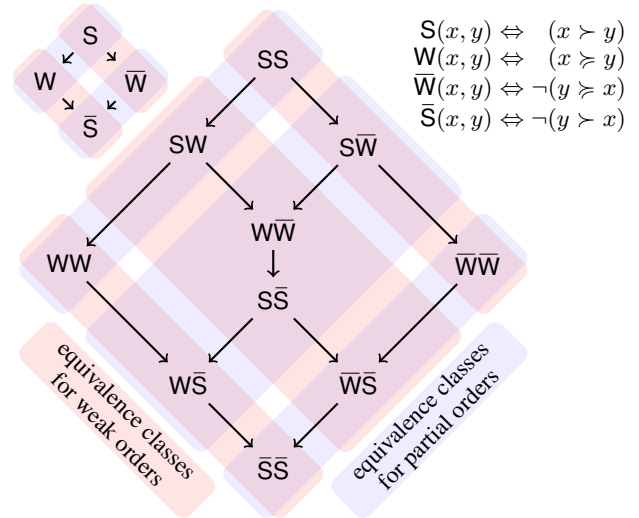


Figure 1: The lattice of single-peakedness conditions. For  $b$  between  $a$  and  $c$ , condition  $XY$  should be read as  $X(b, a) \vee Y(b, c)$ .

### 3.2 Comparison with Related Conditions

Let us now see how our definitions of single-peakedness relate to some of those previously studied in the literature. We are not aware of any such definitions formulated specifically for the general case of preorders, but sometimes existing conditions can easily be adapted to our setting.

But let us first focus on the special case of *weak orders*. Recall (from Figure 1) that when all preferences are weak orders, there are just three distinct notions of single-peakedness:  $SS$ ,  $SW$ , and  $WW$ . Interestingly, for weak orders, they perfectly match three existing notions from the literature, namely those discussed by Fitzsimmons and Lackner [2020]. These authors define a preference to be *necessarily single-peaked* on an axis if every way of breaking ties yields a single-peaked total order on this axis (in Black’s sense). Similarly, a preference is *possibly single-peaked* if there is at least one way of breaking ties such that we obtain a single-peaked total order. Finally, a preference is *single-plateaued* [Moulin, 1984] if the ‘peak’ might contain several tied alternatives but the preference still decreases strictly as we move away from the peak. In the context of weak orders, it is possible to show that necessary single-peakedness coincides with  $SS$ , single-plateauedness with  $SW$ , and possible single-peakedness coincides with  $WW$ .

If we generalise possible single-peakedness, the main concept studied by Fitzsimmons and Lackner [2020], to *all preorders* instead of just weak orders (by considering all ways of completing preorders), it is not anymore equivalent to  $WW$ —or any of our other conditions. To see this, observe that (i) there exists a profile that satisfies  $\bar{W}\bar{W}$ -single-peakedness but not possible single-peakedness and (ii) there exists a profile that satisfies possible single-peakedness but not  $\bar{W}\bar{S}$ -single-peakedness. For part (i), consider a profile of total preferences on six alternatives that is single-peaked on axis  $\triangleright = abcdef$  but no other axis.<sup>1</sup> Add one voter  $i$  with prefer-

<sup>1</sup>The profile containing one voter with preference  $a \succ b \succ c \succ$

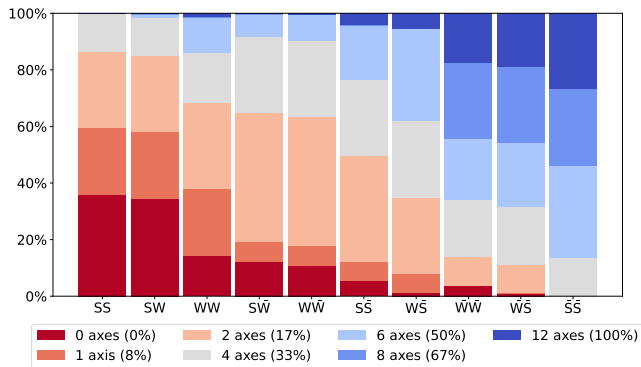


Figure 2: Distribution of the number of axes compatible with a given preorder for four alternatives, for each notion of single-peakedness.

ence  $\{b \succ_i a, b \succ_i c, e \succ_i d, e \succ_i f\}$  to this profile. The resulting profile is  $WW$ -single-peaked on  $\triangleright$ , since the new voter does not violate the condition for any triple. However, no matter how we complete voter  $i$ 's preference, the resulting profile will not be single-peaked on  $\triangleright$ , as there will be two peaks, one at  $b$  and one at  $e$ . Since the original profile was only single-peaked on axis  $\triangleright$ , the completed profile will not be single-peaked on any axis. Thus, the profile is  $WW$ -single-peaked but not possibly single-peaked. For part (ii), take a profile on three alternatives of total preferences such that the only axis on which it is single-peaked is  $\triangleright = abc$  and add one voter  $i$  with preference  $\{c \succ_i b\}$ . The resulting profile is possibly single-peaked but not  $WS$ -single-peaked, as for the new voter we do not have either  $(b \succ_i a)$  or  $\neg(c \succ_i b)$ .

### 3.3 Strength of Conditions

To get a better idea of the relative strengths of the different notions of single-peakedness, we generated all possible preorders for  $m = 4$  alternatives (355 in total) and computed for each of them the number of axes with respect to which it is single-peaked, for each of our ten notions of single-peakedness.<sup>2</sup> The results are shown in Figure 2. At one extreme, for the strongest notion ( $SS$ ), all preorders are compatible with at most one third of the axes, and more than 30% of the preorders are not compatible with any axis at all. At the other extreme, for the weakest notion ( $S\bar{S}$ ), all preorders are compatible with at least one third of the axes, and more than 20% of the preorders are compatible with all axes. Note that any full ranking is single-peaked with respect to exactly one third of the axes, regardless of the notion of single-peakedness considered.

We also counted the number of preferences compatible with a given axis for four alternatives, for each of our notions of single-peakedness. The results are included in the supplementary material and confirm the trend observed in Figure 2.

## 4 Recognising Single-Peakedness

In this section we study the computational complexity of the algorithmic problem of recognising whether a given profile of

$d \succ e \succ f$  and one voter with the reverse order is such a profile.

<sup>2</sup>There are  $m!/2 = 12$  relevant axes (up to axis reversal).

preferences is single-peaked. There are two natural variants of that problem: one where we want to check whether the profile at hand is single-peaked for a specific *given* axis and one where we want to decide whether it is single-peaked for *some* axis. The first problem is easily seen to be tractable.

**Proposition 1.** *For a given profile of preferences and a given axis on the alternatives, deciding whether that profile is single-peaked with respect to that axis is in PTIME, for any of our ten notions of single-peakedness.*

*Proof.* Immediate from the fact that the naïve algorithm that checks the relevant single-peakedness condition for all triples  $(a, b, c)$  with  $\text{Bet}^\triangleright(a, b, c)$  runs in time cubic in the number of alternatives and linear in the number of voters.  $\square$

The other problem is more demanding. We now define it formally, relative to a single-peakedness condition  $C$  (such as  $SW$ ) and a class of preferences  $\mathcal{R}$  (such as the weak orders).

$\exists\text{AXIS}(C, \mathcal{R})$

**Instance:** Profile  $\succ \in \mathcal{R}^n$  of preferences belonging to  $\mathcal{R}$

**Question:** Is  $\succ$  is  $C$ -single-peaked w.r.t. *some* axis?

It is well-known that for the classical model of voting with total orders,  $\exists\text{AXIS}$  can be decided in polynomial time [Bartholdi and Trick, 1986; Doignon and Falmagne, 1994; Escoffier *et al.*, 2008]. Fitzsimmons and Lackner [2020] studied the problem for the notion of possible single-peakedness and found it to be NP-hard for partial orders but polynomial for weak orders. In the remainder of this section we study the complexity of  $\exists\text{AXIS}$  for our setting and find that it depends not only on the class of preferences considered but also on the specific single-peakedness condition we are interested in.

### 4.1 Intractability Results

The problem  $\exists\text{AXIS}$  turns out to be intractable for three of our single-peakedness conditions. As we saw in Section 3.2, all our conditions differ from that of possible single-peakedness studied by Fitzsimmons and Lackner [2020], so their results do not apply here. Still, we can use a very similar proof.

**Theorem 2.**  $\exists\text{AXIS}$  is NP-hard for  $C$ -single-peakedness for any  $C \in \{\bar{S}\bar{S}, \bar{W}\bar{S}, \bar{W}\bar{W}\}$ , even for partial orders.

*Proof.* We prove the claim for  $\bar{S}\bar{S}$ . The other two claims then follow from the fact that  $\bar{S}\bar{S}$ ,  $\bar{W}\bar{S}$ , and  $\bar{W}\bar{W}$  coincide for the special case of partial orders (see Figure 1).

Consider an arbitrary instance of the NP-hard TOTAL ORDERING problem [Opatrny, 1979], where one is given a set of betweenness-statements regarding multiple objects and has to decide whether those objects can be ordered in a way that respects all statements. We interpret the objects involved as alternatives, and for every statement of the form “ $b$  is between  $a$  and  $c$ ” we create two voters, one with preference  $\{b \succ a, c \succ a\}$  and one with preference  $\{a \succ c, b \succ c\}$ . Any axis for which both preferences are  $\bar{S}\bar{S}$ -single-peaked must place  $b$  between  $a$  and  $c$ —and the two preferences constructed do not impose any further requirements on top of this. In particular, declaring incomparability for all other pairs of alternatives is compatible with  $\bar{S}\bar{S}$ . Thus, we get a direct correspondence between potential solutions to the given instance

of TOTAL ORDERING and the constructed instance of  $\exists$ AXIS, meaning that  $\exists$ AXIS must be NP-hard as well.  $\square$

Observe that our proof cannot be adapted to weak orders, because for weak orders we always need to rank all alternatives. Similarly, even for partial orders, it cannot be adapted to the other single-peakedness conditions. For instance, SS is violated if a preference includes too many incomparabilities.

## 4.2 Tractability Results

Not only does our hardness proof not apply to the other conditions, but  $\exists$ AXIS really is tractable in those cases— independently of the class of preferences considered.

**Theorem 3.**  $\exists$ AXIS is in PTIME for  $C$ -single-peakedness for any  $C \in \{\text{SS}, \text{SW}, \text{SW}, \text{WW}, \text{SS}, \text{WW}, \text{WS}\}$ .

*Proof.* We provide a proof for SS. The other proofs are similar (see supplementary material).

The proof works by reduction to the CONSECUTIVE ONES problem [Booth and Lueker, 1976], which is known to be solvable in polynomial time. In that problem, one is given a binary matrix and has to decide whether its columns can be permuted in such a way that, in every row, all 1s appear consecutively. Let  $\succsim$  be a preference profile. We will construct a binary matrix  $M$  such that  $\succsim$  is SS-single-peaked with respect to some axis if and only if  $M$  has the consecutive-ones property. We number the alternatives  $x_1, \dots, x_m$  such that the columns of  $M$  correspond to  $x_1, \dots, x_m$ , in that order. Then, every axis  $\triangleright$  corresponds to exactly one permutation  $\pi$  of the columns of  $M$ , where  $x_j$  is in position  $\pi(j)$  on the axis (assuming an arbitrary left-right orientation of the axis). We denote by  $\pi_\triangleright$  the permutation corresponding to axis  $\triangleright$ .

Informally, for a preorder to be SS-single-peaked with respect to an axis  $\triangleright$ , its ‘peak’ must contain at most two alternatives (that can be tied or incomparable), and moving away from the peak on either side, preferences must always strictly decrease. Let  $\text{TOP}_i = \{a \in X : (b \succ_i a) \text{ for no } b \in X\}$  the set of non-dominated alternatives for voter  $i$ . Then, for every voter  $i$ , we create the following rows in  $M$ :

- **Peak rows:** For all  $a, b \in \text{TOP}_i$  with  $a \neq b$ , create a row  $\mathbf{P}_i[\{a, b\}, \cdot]$  with  $\mathbf{P}_i[\{a, b\}, x] = 1$  iff  $x \in \{a, b\}$ .
- **Upward-closure rows:** For all  $a \in X$ , create a row  $\mathbf{U}_i[a, \cdot]$  with  $\mathbf{U}_i[a, x] = 1$  iff  $x \succ_i a$  or  $x = a$ .

The first type of row ensures that the peak contains at most two alternatives that are next to each other, while the second type of row ensures that preferences strictly decrease as we move away from the peak.

We are done if we can prove that  $\succsim$  is SS-single-peaked with respect to axis  $\triangleright$  if and only if  $M$  has the consecutive-ones property for the corresponding permutation  $\pi_\triangleright$ . This is what we are going to show in the remainder of the proof.

( $\Rightarrow$ ) Assume that  $\succsim$  is SS-single-peaked with respect to  $\triangleright$ . For the sake of contradiction, suppose there exists a row in  $M$  where the 1s are not consecutive after permuting the columns according to  $\pi_\triangleright$ . So there are alternatives  $a, b, c \in X$  with  $\text{Bet}^\triangleright(a, b, c)$ , yet in this row the cells in the columns associated with  $a$  and  $c$  equal 1 and those in the column associated with  $b$  equal 0. We distinguish two cases:

- **Peak row:** Let  $\mathbf{P}_i[\{a, c\}, \cdot]$  be such a row. Since both  $a$  and  $c$  are in  $\text{TOP}_i$ , neither  $b \succ_i a$  nor  $b \succ_i c$  are the case, thereby contradicting SS-single-peakedness.
- **Upward-closure row:** Let  $\mathbf{U}_i[x, \cdot]$  be such a row. We have  $a \succ_i x$  or  $a = x$ , and  $c \succ_i x$  or  $c = x$ , while  $b \not\succ_i x$  and  $b \neq x$ . By transitivity of  $\succ_i$ , this implies that neither  $b \succ_i a$  nor  $b \succ_i c$  can be the case, which contradicts SS-single-peakedness.

( $\Leftarrow$ ) Assume that  $M$  has the consecutive-ones property for permutation  $\pi_\triangleright$ . For the sake of contradiction, suppose there exist a voter  $i$  and alternatives  $a, b, c \in X$  with  $\text{Bet}^\triangleright(a, b, c)$  such that neither  $b \succ_i a$  nor  $b \succ_i c$  holds. Take  $a'$  such that  $a' \in \text{TOP}_i$  and  $a' \succ_i a$  or  $a' = a$  (such an alternative exists by transitivity of  $\succ_i$ ). Note that  $a'$  and  $a$  (if distinct) are on the same side of  $b$  on  $\triangleright$  (that is,  $\neg \text{Bet}^\triangleright(a, b, a')$ ), since otherwise row  $\mathbf{U}_i[a, \cdot]$  would contain a 1 for  $a$  and  $a'$ , but not for  $b$ , contradicting the consecutive-ones property. Similarly, take  $c' \in \text{TOP}_i$  such that  $c' \succ_i c$  or  $c' = c$ . We also have  $\neg \text{Bet}^\triangleright(c, b, c')$ . Since  $a'$  and  $a$  are on the same side of  $b$ , since  $c'$  and  $c$  are on the same side of  $b$ , and since we have  $\text{Bet}^\triangleright(a, b, c)$ , we must have  $\text{Bet}^\triangleright(a', b, c')$ . Now, since  $a', c' \in \text{TOP}_i$ , row  $\mathbf{P}_i[\{a', c'\}, \cdot]$  contains 1s for  $a'$  and  $c'$ , but not for  $b$ , contradicting the consecutive-ones property.  $\square$

Theorems 2 and 3 together provide a complete dichotomy as far as arbitrary preferences and partial orders are concerned. For weak orders, we get tractability for all ten conditions.

**Corollary 4.** For weak orders,  $\exists$ AXIS is in PTIME for any of our ten notions of single-peakedness.

*Proof.* Immediate from Theorem 3, given that for weak orders each of our ten conditions is equivalent to one of the seven conditions covered by that theorem (see Figure 1).  $\square$

For possible single-peakedness Fitzsimmons and Lackner [2020] also establish tractability (using a similar approach).

## 5 Transitivity of Majority Relations

In the classical model of voting, single-peakedness guarantees the existence of an alternative that will not lose a direct majority contest to any other alternative—this is the so-called (weak) *Condorcet winner*. This is a consequence of single-peakedness guaranteeing the transitivity of the majority relation. In this section we explore to what extent we can give similar guarantees for our ten notions of single-peakedness.

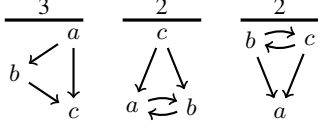
While in the classical model the notion of ‘majority’ is unambiguous, in our model we can distinguish between simple and absolute majorities. We say that  $a$  is preferred by a *simple majority* to  $b$  in profile  $\succsim$  and write  $a \succ_{\text{SM}} b$  in case  $|N_{a>b}^\succsim| > |N_{b>a}^\succsim|$ . And we say that  $a$  is preferred by an *absolute majority* to  $b$  and write  $a \succ_{\text{AM}} b$  in case  $|N_{a>b}^\succsim| > \frac{n}{2}$ .<sup>3</sup>

<sup>3</sup>As an aside, we note that other definitions of absolute majority are possible: we could also count the number of voters who weakly prefer  $a$  to  $b$  or who do not strictly (or weakly) prefer  $b$  to  $a$ . We focus on  $\succ_{\text{AM}}$  as defined here, as it most closely aligns with common usage [Dougherty and Edward, 2010]. Also note that, for the simple majority relation, all of those alternative definitions would coincide.

## 5.1 Simple Majority

Let us first review an example where transitivity of the simple majority relation cannot be guaranteed.

**Example 1.** Consider the following profile with seven voters:



Observe that these are all weak orders that are WW-single-peaked for axes with  $b$  between  $a$  and  $c$ . Yet, the simple majority relation is cyclic, with  $a \succ_{SM} b \succ_{SM} c \succ_{SM} a$ .  $\triangle$

But for the two single-peakedness conditions that are strictly more demanding than WW (see Figure 1) we can show that such a cycle can never occur, as  $\succ_{SM}$  will be transitive.

**Theorem 5.** For any  $C \in \{\text{SS}, \text{SW}\}$ , for any preference profile  $\succ$  that is  $C$ -single-peaked the corresponding simple majority relation  $\succ_{SM}$  is guaranteed to be transitive.

*Proof.* We prove the claim for SW. The claim for SS, which implies SW (see Figure 1), then follows immediately.

Consider any profile  $\succ$  that is SW-single-peaked w.r.t. some axis  $\triangleright$ . Suppose  $a \succ_{SM} b$  and  $b \succ_{SM} c$  (so we need to show  $a \succ_{SM} c$ ). We distinguish three cases, based on how these three alternatives are located on  $\triangleright$ : (1)  $a$  is between  $b$  and  $c$ ; (2)  $b$  is between  $a$  and  $c$ ; (3)  $c$  is between  $a$  and  $b$ .

(1) If  $a$  is between  $b$  and  $c$ , SW requires  $(a \succ_i b) \vee (a \succ_i c)$  and  $(b \succ_i a) \vee (b \succ_i c)$  for every voter  $i$ . By the latter and transitivity of individual preferences,  $b \succ_i c$  implies  $a \succ_i c$ . Similarly,  $c \succ_i a$  implies  $c \succ_i b$ . Now,  $b \succ_{SM} c$  means that strictly more voters  $i$  accept  $b \succ_i c$  than  $c \succ_i b$ . As any voter  $i$  with  $b \succ_i c$  also accepts  $a \succ_i c$ , and any voter  $i$  with  $c \succ_i a$  also accepts  $c \succ_i b$ , strictly more voters accept  $a \succ_i c$  than  $c \succ_i b$ , meaning we get  $a \succ_{SM} c$  as claimed.

(2) If  $b$  is between  $a$  and  $c$ , SW requires  $(b \succ_i a) \vee (b \succ_i c)$  and  $(b \succ_i a) \vee (b \succ_i c)$ . Again, using these conditions and transitivity, we find that  $a \succ_i b$  implies  $a \succ_i c$ ; and  $c \succ_i a$  implies  $b \succ_i a$ . Now,  $a \succ_{SM} b$  means that strictly more voters  $i$  accept  $a \succ_i b$  than  $b \succ_i a$ . Thus, also more voters accept  $a \succ_i c$  than  $c \succ_i a$ , meaning we get  $a \succ_{SM} c$ .

(3) If  $c$  is between  $a$  and  $b$ , SW requires  $(c \succ_i a) \vee (c \succ_i b)$  and  $(c \succ_i a) \vee (c \succ_i b)$ . Then  $b \succ_i c$  implies  $b \succ_i a$ ; and  $a \succ_i b$  implies  $c \succ_i b$ . So we get both  $|N_{b \succ c}^\succ| \leq |N_{b \succ a}^\succ|$  and  $|N_{a \succ b}^\succ| \leq |N_{c \succ b}^\succ|$ . Together with  $a \succ_{SM} b$ , this implies  $|N_{c \succ b}^\succ| > |N_{b \succ c}^\succ|$ , contradicting our assumption  $b \succ_{SM} c$ . So this case cannot occur, meaning the claim always holds.  $\square$

Example 1 demonstrates that Theorem 5 does not generalise to  $C \in \{\text{WW}, \text{WS}, \text{SS}\}$ . Considering the profile of three voters with one reporting the total order  $\{a \succ b, b \succ c, a \succ c\}$  and the other two the partial order  $\{c \succ a\}$  does the same for all remaining conditions. We thus have a complete dichotomy for which single-peakedness conditions can and cannot guarantee transitivity of the simple majority relation. In fact, we also obtain such dichotomies for the special cases of partial orders (only the most demanding condition can guarantee transitivity) and weak orders (only the least demanding condition cannot guarantee transitivity).

## 5.2 Absolute Majority

Let us now consider the absolute majority relation.

**Theorem 6.** For any  $C \in \{\text{SS}, \text{SW}\}$ , for any preference profile  $\succ$  that is  $C$ -single-peaked the corresponding absolute majority relation  $\succ_{AM}$  is guaranteed to be transitive.

The proof is similar to that of Theorem 5 and included in the supplementary material. Example 1 can serve as a counterexample for WW (and the conditions implied by it) also here. But for SW (and the conditions implied by it that are not yet covered by WW) we require a new example. The seven-voter profile in which three voters report  $\{b \succ a, a \succ c, b \succ c\}$ , two report  $\{b \succ a\}$ , and two report  $\{a \succ c\}$  (all of which are partial orders) is SW-single-peaked but gives rise to a cycle in the absolute majority relation. We thus obtain the same three dichotomies as for the simple majority relation.

## 6 Case Studies

When analysing real-world preference data, having multiple notions of single-peakedness at our disposal is useful, and the choice of notion can heavily impact the results obtained. In this section, we illustrate this fact with two case studies.

For both of them, the preferences take the form of top-truncated rankings, which can be interpreted as partial orders, with some alternatives ranked at the top and the rest being incomparable to one another.<sup>4</sup> This leaves three different notions of single-peakedness to consider: SS,  $\overline{\text{SS}}$ , and  $\overline{\overline{\text{SS}}}$ .

Clearly, it is unlikely that real-world preference data will be perfectly single-peaked for any of the notions considered. Therefore, we focus on measuring the *maximal proportion of voters* whose preferences are single-peaked with respect to at least one common axis, similarly to the measure proposed by Niemi [1969]. For each profile and each of the three notions of single-peakedness considered, we compute the axis that maximises this proportion. More demanding notions will typically yield lower maximal proportions.

### 6.1 Scottish Election Data

First, we consider a dataset containing votes cast during Scottish local elections. This dataset was collected by McCune and Graham-Squire [2024] and analysed by Bardal *et al.* [2025]. On this dataset, we observe that SS and  $\overline{\text{SS}}$  always coincide.<sup>5</sup> We grouped the data by the number of alternatives, and for each group, we computed the maximal number of voters whose preferences are single-peaked for the same axis, according to both SS and  $\overline{\text{SS}}$ .

The results are shown in Figure 3. We also highlight the average share of voters with preferences that cannot be single-peaked for *any* axis (which happens for SS), and the average share of voters with preferences that are single-peaked for *all* axes (which happens for  $\overline{\text{SS}}$ ). We observe that the average best score for  $\overline{\text{SS}}$  does not change significantly with the number of alternatives, while the average best score for SS decreases rapidly when the number of alternatives increases,

<sup>4</sup>Alternatively, top-truncated rankings can be interpreted as weak orders, with all unranked alternatives tied at the bottom. This change in interpretation does not affect the findings we report in this section.

<sup>5</sup>This is so because there are no empty preferences in the data.

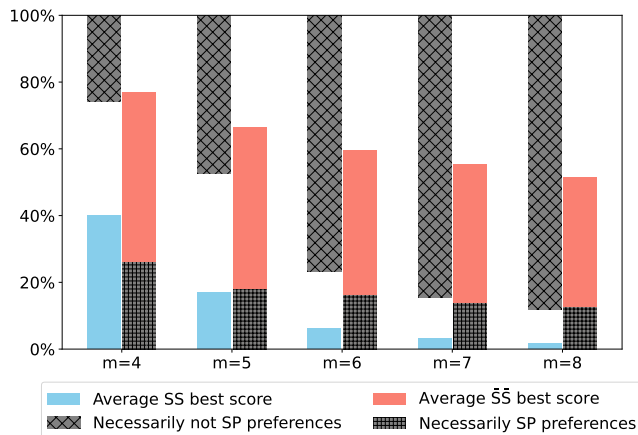


Figure 3: Maximal proportion of voters with single-peaked preferences (over all axes) in Scottish local election data, averaged over all elections with the same number of candidates. Descending histogram bars indicate the share of voters with preferences not single-peaked for any axis (to be read on an inverse scale at  $100\% - y$ ).

mainly due to the increasing share of voters with non-single-peaked preferences for any axis (as any voter who did not rank three or more alternatives cannot satisfy  $\bar{S}\bar{S}$  on any axis).

## 6.2 Deliberative Poll Data

In an influential paper on deliberative polls, List *et al.* [2013] analysed preferences of voters before and after deliberation on a set of policy proposals. They found that in almost all of their datasets, deliberation increases the degree of single-peakedness of participants’ preferences. This would suggest that deliberation is helpful in finding some sort of *meta-consensus*, an axis on which most participants agree. As it is well-known that it is easier find a Condorcet winner when preferences are (approximately) single-peaked, this finding would also suggest that deliberation helps in reaching better and more stable collective decisions.

However, the notion of single-peakedness List *et al.* use corresponds to  $\bar{S}\bar{S}$ , which is the most restrictive notion. We re-analysed their data using the two other relevant notions of single-peakedness,  $\bar{S}\bar{S}$  and  $\bar{S}\bar{S}$ , and the original conclusions of List *et al.* cannot be upheld for these alternative notions. Figure 4 shows the results for one of their datasets, *swepeco*, which contains preferences of participants on energy policies. The results presented by List *et al.* correspond to the left-most bars in the figure, for which we clearly observe an increase in single-peakedness after deliberation. However, note that before deliberation, a large fraction of voters had preferences that were incompatible with  $\bar{S}\bar{S}$  on any axis (as indicated by the top grey bar). This is because before deliberation many participants did not provide any preference at all, which makes it impossible for their preferences to be  $\bar{S}\bar{S}$ -single-peaked. On the other hand, these participants would automatically be considered as satisfying  $\bar{S}\bar{S}$ . Therefore, when considering  $\bar{S}\bar{S}$ , we observe a *decrease* in single-peakedness after deliberation. The decrease is even more pronounced when considering  $\bar{S}\bar{S}$ . As shown in the supplementary material, a similar pattern can be observed for many of the other datasets

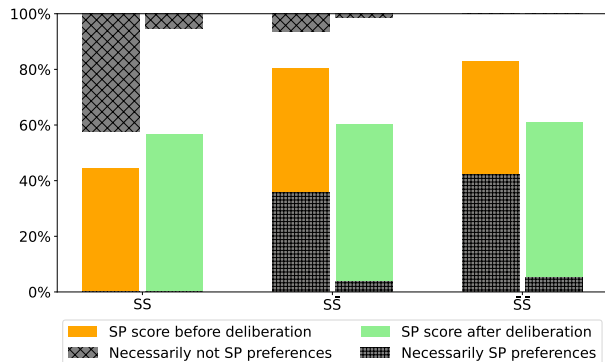


Figure 4: Maximal proportion of voters with single-peaked preferences (over all axes) in the *swepeco* dataset, before and after deliberation with different notions of single-peakedness.

analysed by List *et al.*, suggesting that most of the increase in single-peakedness is actually due to a reduction in the number of participants with incomplete preferences rather than to a genuine increase in agreement on an axis. This example illustrates how the choice of single-peakedness notion can significantly impact the conclusions drawn from real-world data.

## 7 Conclusion

Starting from the classical definition of single-peakedness expressed in terms of *betweenness*, we explored the space of natural generalisations of this definition to profiles of preferences that might include both indifferences and incomparabilities. We identified and systematically studied the ten resulting notions of single-peakedness in algorithmic terms (analysing the complexity of recognising when such a condition is met), in axiomatic terms (analysing when a condition guarantees transitivity of majority preferences), and in practical terms (by applying our conditions to real-world preference data). We found significant differences between alternative definitions along each of these dimensions.

One now might wonder what the ‘right’ definition of single-peakedness is. The answer clearly will depend on context, but one main take-away, especially of our experimental observations, is that having multiple definitions at hand can be useful. Having said this, the three conditions along the ‘southeastern’ diagonal in Figure 1 ( $\bar{S}\bar{S}$ ,  $\bar{W}\bar{S}$ ,  $\bar{W}\bar{W}$ ) arguably are somewhat less attractive. Due to their permissive nature, they are further removed from the original intuitive idea of ‘single-peakedness’, they are NP-hard to recognise, and they cannot guarantee the transitivity of majority relations.

In other work on single-peakedness in computational social choice, the interest is frequently in the computational complexity of outcome determination for voting rules when profiles are single-peaked [Elkind *et al.*, 2025]. This is a natural direction for future work also in our setting—even though for incomplete preferences it is less clear which voting rules one should focus on [Terzopoulou, 2021]. Furthermore, it would be interesting to explore notions of *approximate* single-peakedness for our setting [Tydrichová, 2023].

## References

- [Arrow *et al.*, 2002] Kenneth J. Arrow, Amartya K. Sen, and Kotaro Suzumura, editors. *Handbook of Social Choice and Welfare*, volume 1. North-Holland, 2002.
- [Bardal *et al.*, 2025] Tuva Bardal, Markus Brill, David McCune, and Jannik Peters. Proportional representation in practice: Quantifying proportionality in ordinal elections. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence*, 2025.
- [Bartholdi and Trick, 1986] John Bartholdi, III and Michael A. Trick. Stable matching with preferences derived from a psychological model. *Operations Research Letters*, 5(4):165–169, 1986.
- [Black, 1948] Duncan Black. On the rationale of group decision-making. *Journal of Political Economy*, 56(1):23–34, 1948.
- [Booth and Lueker, 1976] Kellogg S. Booth and George S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *Journal of Computer and System Sciences*, 13(3):335–379, 1976.
- [Brandt *et al.*, 2016] Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors. *Handbook of Computational Social Choice*. Cambridge University Press, 2016.
- [Doignon and Falmagne, 1994] Jean-Paul Doignon and Jean-Claude Falmagne. A polynomial time algorithm for unidimensional unfolding representations. *Journal of Algorithms*, 16(2):218–233, 1994.
- [Dougherty and Edward, 2010] Keith L. Dougherty and Julian Edward. The properties of simple vs. absolute majority rule: Cases where absences and abstentions are important. *Journal of Theoretical Politics*, 22(1):85–122, 2010.
- [Elkind and Lackner, 2015] Edith Elkind and Martin Lackner. Structure in dichotomous preferences. In *Proceedings of the 24th International Joint Conference on Artificial Intelligence (IJCAI)*, 2015.
- [Elkind *et al.*, 2025] Edith Elkind, Martin Lackner, and Dominik Peters. Preference restrictions in computational social choice: A survey. arXiv:2205.09092, 2025.
- [Escoffier *et al.*, 2008] Bruno Escoffier, Jérôme Lang, and Meltem Öztürk. Single-peaked consistency and its complexity. In *Proceedings of the 18th European Conference on Artificial Intelligence (ECAI)*, 2008.
- [Farrar *et al.*, 2010] Cynthia Farrar, James S. Fishkin, Donald P. Green, Christian List, Robert C. Luskin, and Elizabeth Levy Paluck. Disaggregating deliberation’s effects: An experiment within a deliberative poll. *British Journal of Political Science*, 40(2):333–347, 2010.
- [Feld and Grofman, 1986] Scott L. Feld and Bernard Grofman. Partial single-peakedness: An extension and clarification. *Public Choice*, 51(1):71–80, 1986.
- [Fitzsimmons and Lackner, 2020] Zack Fitzsimmons and Martin Lackner. Incomplete preferences in single-peaked electorates. *Journal of Artificial Intelligence Research*, 67:797–833, 2020.
- [Gaertner, 2001] Wulf Gaertner. *Domain Conditions in Social Choice Theory*. Cambridge University Press, 2001.
- [Grossi *et al.*, 2024] Davide Grossi, Ulrike Hahn, Michael Mäs, Andreas Nitsche, Jan Behrens, Niclas Boehmer, Markus Brill, Ulle Endriss, Umberto Grandi, Adrian Haret, Jobst Heitzig, Nicolien Janssens, Catholijn M. Jonker, Marijn A. Keijzer, Axel Kistner, Martin Lackner, Alexandra Lieben, Anna Mikhaylovskaya, Pradeep K. Murukannaiah, Carlo Proietti, Manon Revel, Élise Rouméas, Ehud Shapiro, Gogulapati Sreedurga, Björn Swierczek, Nimrod Talmon, Paolo Turrini, Zoi Terzopoulou, and Frederik Van De Putte. Enabling the digital democratic revival: A research program for digital democracy. arXiv:2401.16863, 2024.
- [Lang, 2020] Jérôme Lang. Collective decision making under incomplete knowledge: Possible and necessary solutions. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI)*, 2020.
- [List *et al.*, 2013] Christian List, Robert C. Luskin, James S. Fishkin, and Iain McLean. Deliberation, single-peakedness, and the possibility of meaningful democracy: Evidence from deliberative polls. *The Journal of Politics*, 75(1):80–95, 2013.
- [McCune and Graham-Squire, 2024] David McCune and Adam Graham-Squire. Monotonicity anomalies in Scottish local government elections. *Social Choice and Welfare*, 63(1):69–101, 2024.
- [Moulin, 1984] Hervé Moulin. Generalized Condorcet-winners for single peaked and single-plateau preferences. *Social Choice and Welfare*, 1(2):127–147, 1984.
- [Niemi, 1969] Richard G. Niemi. Majority decision-making with partial unidimensionality. *American Political Science Review*, 63(2):488–497, 1969.
- [Opatrny, 1979] Jaroslav Opatrny. Total ordering problem. *SIAM Journal on Computing*, 8(1):111–114, 1979.
- [Rad and Roy, 2021] Soroush Rafiee Rad and Olivier Roy. Deliberation, single-peakedness, and coherent aggregation. *American Political Science Review*, 115(2):629–648, 2021.
- [Terzopoulou, 2021] Zoi Terzopoulou. *Collective Decisions with Incomplete Individual Opinions*. PhD thesis, University of Amsterdam, 2021.
- [Tydrichová, 2023] Magdaléna Tydrichová. *Structural and Algorithmic Aspects of Preference Domain Restrictions in Collective Decision Making: Contributions to the Study of Single-Peaked and Euclidean Preferences*. PhD thesis, Sorbonne Université, 2023.
- [Zwicker, 2016] William S. Zwicker. Introduction to voting theory. In Felix Brandt, Vincent Conitzer, Ulle Endriss, Jérôme Lang, and Ariel D. Procaccia, editors, *Handbook of Computational Social Choice*, chapter 2, pages 23–56. Cambridge University Press, 2016.