

Measuring a Priori Voting Power - Taking Delegations Seriously

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Abstract

We introduce new power indices to measure the a priori voting power of voters in liquid democracy elections where an underlying network restricts delegations. We argue that our power indices are natural extensions of the standard Penrose-Banzhaf index in simple voting games. We show that computing the criticality of a voter is #P-hard even when voting weights are polynomially-bounded in the size of the instance. However, for specific settings, such as when the underlying network is a bipartite or complete graph, recursive formulas can compute these indices for weighted voting games in pseudo-polynomial time. We highlight their theoretical properties and provide numerical results to illustrate how restricting the possible delegations can alter voters' voting power.

1 Introduction

Voting games have been used extensively to study the a priori voting power of voters participating in an election [Felsenthal *et al.*, 1998]. A priori voting power means the power granted solely by the rules governing the election process. Notably, these measures do not consider the nature of the bill nor the affinities between voters. The class of I-power measures (where I represents influence) “calculate” how likely a voter will influence the outcome. Several I-power measures have been defined, the best known being the Penrose-Banzhaf measure in simple voting games [Banzhaf III, 1964; Penrose, 1946]. In simple voting games, an assembly of voters make a collective decision on a proposal which voters either support or oppose. The Penrose-Banzhaf measure is as follows: voters are assumed to vote independently from one another; a voter is as likely to vote in favour or against the proposal. It then measures the probability of a voter altering the election's outcome given this probabilistic model.

Simple voting games have been extended in several directions to take into account more complex and realistic frameworks. For example, taking into account abstention [Freixas, 2012], several levels of approval [Freixas and Zwicker, 2003], or coalition structures [Owen, 1981]. Hence, new power indices have been designed to better understand voters’ criticality in these frameworks. However, elections with delegations have been largely unexplored with respect to a priori voting power. Yet, frameworks such as Proxy Voting (PV) [Miller, 1969; Tullock, 1992] or Liquid Democracy (LD) [Behrens *et al.*, 2014; Brill, 2018] have received increasing interest in the AI community due to their ability to provide more flexibility and engagement in the voting process. Thus, studying these frameworks via their distribution of a priori voting power is an interesting research direction.

Our contribution. We extend simple voting games to model elections where voters can delegate their votes through a social network, modelled as a digraph G . Our model encapsulates both the LD and PV settings. We design a new I-voting power measure to measure voters’ criticality in these settings. We argue that our power measure is a natural extension of the Penrose-Banzhaf measure, and we illustrate the intuitions behind it through various examples. When G is an arbitrary digraph, we show that the computation of our measure is $\#P$ -hard even when voters’ weights are polynomially-bounded in the number of voters. However, we prove that it can be computed for weighted voting games in pseudo-polynomial time in the PV setting, in which the graph G is directed bipartite with all arcs going from one side (possible delegators) to the other (proxies), and in the LD setting when G is complete. Last, we complement our theoretical results with numerical results to illustrate how introducing delegations modifies voters’ a priori voting power.

Related Work

Voting power. Measuring a voter’s voting power in a specific setting quantifies how *critical* they are in deciding the outcome of the election. A voter i ’s voting power can be considered as the difference in probability of i voting for the issue when the outcome is also in favour and i voting for the issue when the outcome is not [Gelman *et al.*, 2002]. We give an overview of some standard measures (we recommend [Lucas, 1974] for an overview of voting power and [Felsenthal and Machover, 2005] for a historical overview).

The measure introduced by Shapley and Shubik [1954] quantifies the voter’s expected pay-off, known as P-power, unlike the other measures we will discuss. P-power differs in motivation from I-power as P-power cares about the voter being part of the winning coalition, sharing the coalition’s utility among its members, with those not in the coalition receiving utility 0. In contrast, I-power has a policy-seeking motivation and is, therefore, concerned with the voter’s stance on the issue.

I-power was independently given a mathematical explanation by Penrose [1946], Banzhaf III [1964], and Coleman [1971]. It counts, for an agent i , in how many of the 2^n possible voting profiles that changing i ’s vote from 0 to 1 changes the

outcome. The *Banzhaf measure* (or absolute Banzhaf index) is denoted by β' , whereas the *Banzhaf index* is the relative quantity denoted by β (found from normalising β').

Extending the notion of voting power. Standard voting power measures are defined on binary issues. Yet, as the study of voting models has advanced, so has the study of voting power. One generalisation is to the domain of the available votes, thus, moving away from binary decisions on the issue. Influenced by Felsenthal and Machover [2001], probabilistic models of voting power with abstention and a binary outcome are well-studied [Felsenthal and Machover, 1997; Freixas, 2012]. Freixas and Lucchetti [2016] extended the Banzhaf index by introducing two measures of being positively critical, i.e., changing your vote from being for the issue to abstaining and an abstaining vote to be against the issue. Voting games with approvals form a subclass of voting games with varying levels of approvals in both the input and output of the election [Freixas and Zwicker, 2003]. Another well-studied extension of the standard notions of voting power measures in WVGs is to allow for randomised weights. The Shapley-Shubik index has been well-studied in these settings [Filmus *et al.*, 2016; Bachrach *et al.*, 2016]. Boratyn *et al.* [2020] also studied the Banzhaf index in this setting; this is close to our own when focussing on proxy voting elections.

Proxy voting (PV) and liquid democracy (LD). PV allows agents to choose their proxy from a list of representatives who will vote on their behalf. In some models, a delegator may only choose a proxy from the list of representatives [Abramowitz and Mattei, 2019; Alger, 2006; Cohensius *et al.*, 2017]. In other models, delegators can also vote directly yet still cannot receive votes [Green-Armytage, 2015; Miller, 1969; Tullock, 1992].

Models of LD allow voters to either vote on the issue or delegate their vote to another voter, which can be transitively delegated further. Some examples of recent advancements in the study of LD are: extending the model to account for different situations, whether it be ranked delegations [Brill *et al.*, 2021; Colley *et al.*, 2022; Kotsialou and Riley, 2020] or allowing for multiple interconnected issues [Brill and Talmon, 2018; Jain *et al.*, 2021]; assessing how successful LD is in finding a ground truth [Halpern *et al.*, 2021; Kahng *et al.*, 2018]; or studying (non-cooperative) game-theoretic aspects [Bloembergen *et al.*, 2019; Escoffier *et al.*, 2020; Markakis and Papatotiropoulos, 2021; Noel *et al.*, 2021].

Our closest work is that of Zhang and Grossi [2021], who study a version of the Banzhaf measure in LD. Their measure, for a given delegation graph, determines how critical an agent is in changing the outcome. Our work differs as we focus on *a priori* voting power, where no prior knowledge is known about the election, such as a specific delegation graph.¹

¹In Appendix A.2, we give a probabilistic model where the Banzhaf measure from Zhang and Grossi [2021] can be interpreted as an I-power measure.

2 Model

Let V be a set of n voters taking part in an election to decide if some binary proposal should be accepted or not. Each voter has different possible actions: they may vote directly, either for (1) or against (-1) the proposal or delegate their vote to another voter. A voter who decides to vote (resp. delegate) will be termed a delegatee (resp. delegator). An underlying social network $G = (V, E)$ restricts the possible delegations between the agents, hence voter $i \in V$ can only delegate to a voter in their out-neighbourhood $\text{NB}_{out}(i) = \{j \in V \mid (i, j) \in E\}$. We will consider in more detail two cases: when G is complete and when G is bipartite, where the former corresponds to the LD setting when voters can choose any other voter as a delegate and the latter corresponds to PV.

Definition 1. *Given a digraph $G = (V, E)$, a G -delegation partition D is a map defined on V such that $D(i) \in \text{NB}_{out}(i) \cup \{-1, 1\}$ for all $i \in V$. We let \mathcal{D} be the set of all such partitions and D^- , D^+ , and D^v be the inverse images of $\{-1\}$, $\{1\}$ and $\{v\}$ for each $v \in V$ under D .*

Whereas a direct-vote partition divides the voters such that each partition cell corresponds to a possible voting option. We allow for abstentions to model situations in which a delegator does not have a delegatee voting on their behalf (e.g., due to delegation cycles).

Definition 2. *A direct-vote partition of a set V is a map T from V to the votes $\{-1, 0, 1\}$. We let T^- , T^0 , and T^+ denote the inverse images of $\{-1\}$, $\{0\}$ and $\{1\}$ under T .*

A G -delegation partition D naturally induces a direct-vote partition T_D by resolving the delegations. First, we let voters in D^- , and D^+ also be in T^- , and T^+ , respectively. From this point, for some $\circ \in \{-, +\}$, if $v' \in D^v$ and $v \in T^\circ$, then $v' \in T^\circ$. This continues until no more voters can be added to T^+ or T^- . The remaining unassigned agents abstain and thus are in T^0 . This procedure assigns agents their delegate's vote unless it leads to a cycle, in this case, their vote is recorded as an abstention.

Next we define a partial ordering \leq among direct-vote partitions: if T_1 and T_2 are two direct-vote partitions of V , we let: $T_1 \leq T_2 \Leftrightarrow T_1(x) \leq T_2(x), \forall x \in V$.

Definition 3. *A ternary (resp. binary) voting rule is a map W from the set $\{-1, 0, 1\}^n$ (resp. $\{-1, 1\}^n$) of all direct-vote partitions (resp. all direct-vote partitions without abstention) of V to $\{-1, 1\}$ satisfying the following conditions:*

1. $W(\mathbb{1}) = 1$ and $W(-\mathbb{1}) = -1$ where $\mathbb{1} = \underbrace{(1, \dots, 1)}_{\times n}$;

2. Monotonicity: $T_1 \leq T_2 \Rightarrow W(T_1) \leq W(T_2)$.²

²All ternary voting rules do not satisfy monotonicity, e.g., a weighted voting rule with an additional quorum condition. However, we enforce this condition such that we may only look at the election result when the voter favours the proposal on the one hand and against the proposal on the other to define criticality.

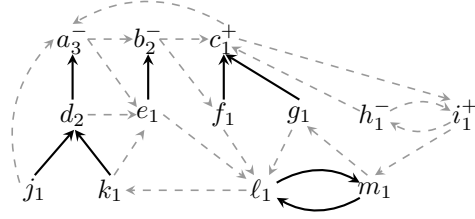


Figure 1: The underlying network G used in Example 1. While all edges give us E , the solid edges give us a valid G -delegation partition where the superscripts of $+$ or $-$ represent delegates direct votes. Each node's subscript refers to its voting weight.

Note that ternary (and binary) voting rules only use the direct-vote partition to find an outcome, i.e., only the information of which agents voted directly or indirectly in favour or against the proposal or abstained. Thus, these rules do not need the delegations to find an outcome.

A ternary (resp. binary) voting rule is *symmetric* if $W(T) = -W(-T)$, where $-T$ is the direct-vote partition defined by $-T(x) = -(T(x))$, $\forall x \in V$. Moreover, for ease of notation, we may also use $W(T^+, T^-)$ to denote $W(T)$, noting that T^0 can be obtained from T^+ and T^- .

Weighted voting games. Weighted Voting Games (WVGs) express ternary voting rules compactly, with a quota $q \in (0.5, 1]$ and a map $w : V \rightarrow \mathbb{N}_{>0}$ assigning each voter a positive weight. Given a set $S \subseteq V$, we let $w(S) = \sum_{i \in S} w(i)$. In a WVG with weight function w , we let $W(T) = 1$ iff $w(T^+) > q \times w(T^+ \cup T^-)$, i.e., the proposal is accepted if the sum of the voters' weights for the proposal is greater than a proportion q of the total weight of non-abstaining voters; otherwise, the proposal is rejected.

Example 1. Consider agents $V = \{a, b, \dots, m\}$ connected by the underlying network G as depicted in Figure 1. The solid lines give a valid G -delegation partition D with $D^+ = \{c, i\}$, $D^- = \{a, b, h\}$, $D^a = \{d\}$, $D^b = \{e\}$, $D^c = \{f, g\}$, $D^d = \{j, k\}$, $D^e = \{m\}$, and $D^m = \{\ell\}$. This G -delegation partition induces the following direct-vote partition: $T^+ = \{c, f, g, i\}$, $T^- = \{a, b, d, e, h, j, k\}$, and $T^0 = \{\ell, m\}$. Consider W induced by the WVG where $q = 0.5$ and $w(a) = 3$, $w(b) = w(d) = 2$ and the remaining voters $x \in V \setminus \{a, b, d\}$ have weight $w(x) = 1$. The proposal is rejected under this G -delegation partition as $w(T^+) = 4 \leq 7.5 = q \cdot w(T^+ \cup T^-) = 15/2$.

We conclude this subsection with some notation. Given a set X , let $\mathcal{P}_k(X)$ denote the set of k ordered partitions of X . By ordered partitions, we mean that $(\{1\}, \{2, 3\})$ should be considered different to $(\{2, 3\}, \{1\})$. Next, given a voting rule W , a voter $i \in V$, and (X, Y, Z) three non-intersecting subsets of $V \setminus \{i\}$, we define:

$$\delta_{i, \rightarrow +}^W(X, Y, Z) = \frac{W(S \cup U \cup \{i\}, T) - W(X, Y \cup Z \cup \{i\})}{2}.$$

We say a voter $i \in V$ is critical when they can affect the outcome of the vote. Thus for three non-intersecting subsets of $V \setminus \{i\}$, namely X, Y, Z , where X (resp. Y) denotes the set of voters opposing (resp. supporting) the proposal through their vote of delegation, and Z is the set of voters delegating directly or not to i , then i is critical if and only if $\delta_{i, \rightarrow+}^W(X, Y, Z) > 0$. We say a voter $i \in V$ is *positively* (resp. *negatively*) critical if by changing a positive (resp. negative) vote to a negative (resp. positive) one, the outcome will also change from being for to against (resp. against to for) the issue. In Example 1, we see that a is critical in this G -delegation partition, as $V \setminus \{a\}$ is partitioned as such $X = \{c, f, g, i\}$, $Y = \{b, e, h\}$ and $Z = \{d, j, k\}$ and thus $\delta_{a, \rightarrow+}^W(X, Y, Z) = \frac{1 - (-1)}{2} = 1$.

2.1 Modelling a priori voting power

We aim to measure *a priori* voting power in this setting. An agent's voting power is their probability of being able to affect the election's outcome. Similarly to the intuitions behind the Penrose-Banzhaf measure, we invoke the principle of insufficient reason. There are two ways of seeing this principle.

The global uniformity assumption. If there is no information about the proposal or voters, we assume all G -delegation partitions are equally likely with probability $\prod_{i \in V} \frac{1}{|\text{NB}_{out}(i)|+2}$. In Example 1, as $|\text{NB}_{out}(i)| = 2$ for every $i \in V$, this means that every G -delegation partition occurs with probability $(\frac{1}{4})^{13}$.

The individual uniformity assumption. The global uniformity assumption is similar to a model in which each voter delegates with probability $p_d^i = \frac{1}{|\text{NB}_{out}(i)|+2}$ and votes with probability $p_v^i = 1 - p_d^i = \frac{2}{|\text{NB}_{out}(i)|+2}$. Delegation (resp. voting) options are chosen uniformly at random and voters make their choices independently from one another. This is consistent with the idea that we have no information about voters' personalities and interests, or the nature of the proposal. Hence, voters should be equally likely to support (probability p_y) or oppose (probability p_n) the proposal, i.e., $p_y = p_n = 1/2$. Moreover, in ignorance of any concurrence or opposition of interests between voters, we should assume that the likelihood of a voter choosing between each of their possible delegates is equally likely, i.e., the probability that a delegator i delegates to a voter $j \in \text{NB}_{out}(i)$ is $1/|\text{NB}_{out}(i)|$. The *individual uniformity assumption* is an extension of the global uniformity assumption in which p_d^i can be any value in $[0, 1]$ dependent on $|\text{NB}_{out}(i)|$, such that $p_d^i = 0$ when $\text{NB}_{out}(i) = \emptyset$.

For generality, we consider this latter model unless specified otherwise. We now define the LD Penrose-Banzhaf measure of a voter i for a given underlying graph G when considering that the probability of each G -delegation partition is determined by the individual uniformity assumption.

Definition 4 (LD Penrose-Banzhaf measure). *Given a digraph $G = (V, E)$ and a ternary voting rule W , the LD Penrose-Banzhaf measure of voter $i \in V$ is*

defined as:

$$\mathcal{M}_i^{ld}(W, G) = \sum_{D \in \mathcal{D}} \mathbb{P}(D) \frac{W(T_{D_i^+}) - W(T_{D_i^-})}{2},$$

where $\mathbb{P}(D)$ is the probability of the G -delegation partition D occurring, and D_i^+ (resp. D_i^-) is the G -delegation partition identical to D with the only possible difference being that i supports (resp. opposes) the proposal.

\mathcal{M}_i^{ld} quantifies the probability to sample a delegation partition where i is able to alter the election's outcome (formally stated in the following Theorem).

Theorem 1. *Given a digraph $G = (V, E)$, a ternary voting rule W , and a voter $i \in V$, we have that:*

$$\mathbb{P}(i \text{ is critical}) = \mathcal{M}_i^{ld}(W, G).$$

Moreover, if $\text{NB}_{out}(i) = \emptyset$ or W is symmetrical, we have that:

$$\begin{aligned} \mathbb{P}(i \text{ is positively critical}) &= \mathcal{M}_i^{ld}(W, G)/2 \\ &= \mathbb{P}(i \text{ is negatively critical}). \end{aligned}$$

This proof relies on the fact that we are summing over the probability of each D with respect to $W(T_{D_i^+}) - W(T_{D_i^-})$, which measures when the voter i is critical. Recall that being positively critical means that changing a vote for to against the issue will also change the outcome in the same way (negatively critical is defined similarly). Furthermore, this happens equally when $\text{NB}_{out}(i) = \emptyset$ (the only option is to vote either for or against the issue) or when W is symmetric.

For the second part of Theorem 1, the condition is necessary as if W reflects unanimity, i.e., $W(T) = 1$ iff $T = \mathbb{1}$, then voters will be more likely to be positively critical than negatively critical.³ Additionally, observe that the LD Penrose-Banzhaf measure of voting power extends the standard Penrose-Banzhaf measure (formalized in Proposition 1) and that its values are not normalized (i.e., summing over the agents does not yield 1). The corresponding voting power index can be defined by normalizing over voters.

Proposition 1. *If $p_d^i = 0$ for all $i \in V$, e.g., if $E = \emptyset$, then the LD Penrose-Banzhaf measure of voting power is equivalent to the standard Penrose-Banzhaf voting power measure.*

³If the voting rule requires total agreement to accept the proposal, then voter i will be critical iff all other voters agree on the proposal. Thus, the probability that i is critical while voting directly or indirectly in favour of the proposal is higher than i being critical while voting directly or indirectly against the proposal.

3 Hardness of computation

Computing the standard Penrose-Banzhaf measure in WVGs is $\#P$ -complete [Prasad and Kelly, 1990]. However, it can be computed by a pseudo-polynomial algorithm that runs in polynomial time w.r.t. the number of voters and the maximum weight of a voter [Matsui and Matsui, 2000]. We show that the problem of computing the *LD* Penrose-Banzhaf measure is $\#P$ -hard even when voter’s weights are bounded linearly by the number of voters. Hence, a similar pseudo-polynomial algorithm is unlikely to exist. The proof uses an enumeration trick inspired by that of Chen *et al.* [2010, Theorem 1]. Informally speaking, this trick shows that one can solve the $\#P$ -hard problem of counting the number of simple paths between two vertices in a digraph by using a polynomial number of calls to a subroutine solving our power measure computation problem, and then inverting a specific Vandermonde matrix. Hence, note that the type of reduction that is used is a Turing reduction.

Theorem 2. *Given a digraph $G = (V, E)$ and a WVG defined on V , computing the LD-Penrose-Banzhaf power measure of a voter is $\#P$ -hard under Turing reductions even when voter’s weights are bounded linearly by the number of voters.*

Proof sketch. We give a reduction from the problem of counting simple paths in a digraph which is known to be $\#P$ -complete [Valiant, 1979]. The problem takes as input a digraph $G = (V, E)$ and nodes $s, t \in V$. The problem then returns the number of simple paths from s to t in G . Let denote \mathcal{P}_ℓ the set of paths of length ℓ between s and t in G . Given $G = (V, E)$ and $s, t \in V$ two vertices, we create $|V| + 1$ different digraphs $G_k = (V_k, E_k)$ with $k \in \{0, \dots, |V| + 1\}$ such that G_k is obtained by modifying G to impose some condition on the out-degree of nodes in V . Thus, in each digraph $G_k = (V_k, E_k)$, we consider a WVG where weights are linearly bounded in $|V|$ and such that voter s is a dictator. Hence, t is only critical when in s ’s delegation path. Under the individual uniformity assumption, we obtain the criticality of t in each G_k as a weighted sum of values $|\mathcal{P}_\ell|$ such that the weights of these $|V| + 1$ equations form a Vandermonde matrix. Inverting this matrix makes it possible to derive the values $|\mathcal{P}_\ell|$ from the criticality of t in each graph G_k , and thus to solve the problem of counting simple paths from s to t . \square

While computing exactly the *LD* Penrose-Banzhaf measures of voting power is hard, these values can be approximated easily using a standard sampling procedure. We sample enough G -delegation partitions by simulating the behaviours of the different voters according to the individual uniformity assumption and consider the expected criticality of the voters given these samples. Relying on Hoeffding’s inequality, one can then prove that these estimates are within some ϵ of the true voting power measure.

In the next two sections, we discuss two restricted classes of instances for which more compact formulations of the *LD* Penrose-Banzhaf measure can be designed such that there exists a pseudo-polynomial algorithm.

4 Proxy voting

This section models a PV setting where $G = (V, E)$ is bipartite with $V = (V_d, V_v)$ and $E = \{(i, j) \mid i \in V_d, j \in V_v\}$. The set of delegates V_v is given in input and is predetermined, e.g., by an election, self-nomination, or sortition. Each delegatee $i \in V_v$ will vote, i.e., $\text{NB}_{out}(i) = \emptyset$ and $p_d = 0$, whereas each voter $i \in V_d$ can vote or delegate to any delegatee in V_v , i.e., $\text{NB}_{out}(i) = V_v$.⁴ Note that, under our individual uniformity assumption, the probability of delegating for each $i \in V_d$ is equal as they all have the same out-degree. We denote this value by p_d and let $p_v = 1 - p_d$. Moreover, let $n_v = |V_v|$ and $n_d = |V_d| = n - n_v$.

We provide more compact formulas for the LD Penrose-Banzhaf measure in this PV setting. We only consider binary voting rules as there cannot be delegation cycles in this setting ($T^0 = \emptyset$).

To measure how critical an agent i can be, we consider partitions of $V \setminus \{i\}$ into three sets V^+, V^-, V^i where V^+ (resp. V^-) represents the n^+ (resp. n^-) voters whose final vote is in favour of (resp. against) the proposal, either by voting directly or indirectly and V^i is the set of n^i voters who delegate to voter i . Note that V^+, V^-, V^i form a partition of $V \setminus \{i\}$ and $V^i = \emptyset$ when $i \in V_d$. We focus on how these sets intersect V_d and V_v . We define V_d^+, V_d^-, V_v^+ , and V_v^- with size n_d^+, n_d^-, n_v^+ , and n_v^- , respectively, such that $V_x^\circ = V_x \cap V^\circ$ for $x \in \{v, d\}$ and $\circ \in \{-, +\}$.

Given our probabilistic model of delegation partitions, observe that the probability of having a partition V^+, V^-, V^i only depends on these cardinalities. More precisely:

– When $i \in V_v$, note that $n_v^- = n_v - 1 - n_v^+$ and $n^i = n_d - n_d^+ - n_d^-$. Hence, we denote this probability of having such a partition V^+, V^-, V^i by $P_v(n_v^+, n_d^+, n_d^-)$:

$$P_v(n_v^+, n_d^+, n_d^-) = \frac{1}{2^{n_v-1}} \left(\frac{p_v}{2} + p_d \frac{n_v^+}{n_v} \right)^{n_d^+} \times \left(\frac{p_v}{2} + p_d \frac{n_v^-}{n_v} \right)^{n_d^-} \left(\frac{p_d}{n_v} \right)^{n^i}. \quad (1)$$

– When $i \in V_d$, note that $n_v^- = n_v - n_v^+$ and $n_d^- = n_d - 1 - n_d^+$. We let $P_d(n_v^+, n_d^+)$ denote the probability of having such a partition of V^+, V^- :

$$P_d(n_v^+, n_d^+) = \frac{1}{2^{n_v}} \left(\frac{p_v}{2} + p_d \frac{n_v^+}{n_v} \right)^{n_d^+} \left(\frac{p_v}{2} + p_d \frac{n_v^-}{n_v} \right)^{n_d^-}. \quad (2)$$

There are some conditions on the integer parameters n_v^+, n_d^+ , and n_d^- . If $i \in V_v$, we have that $n_v^+ \leq n_v - 1$, and $n_d^+ + n_d^- \leq n_d$. If $i \in V_d$, we have $n_v^+ \leq n_v$, and $n_d^+ + n_d^- = n_d - 1$. If these conditions are not respected, we set $P_v(n_v^+, n_d^+, n_d^-) = 0$ (resp. $P_d(n_v^+, n_d^+) = 0$).

We now detail Equation 1. Equation 2 is obtained similarly. The probability of the binary votes of the delegates other than i being a certain way is $(1/2)^{n_v-1}$.

⁴We present an alternative PV model in Appendix A.4 where voters in V_d must delegate to a voter in V_v . We show that voters' criticalities can also be computed by a pseudo-polynomial algorithm.

Then, the probability that each voter in V_d^+ (resp. V_d^-) votes in favour of the proposal is $p_v/2 + p_d n_v^+/n_v$ (resp. against is $p_v/2 + p_d n_v^-/n_v$) where the first summand corresponds to the case in which the voter votes and the second to the one in which they delegate. Last, the probability that each voter in V_d^i delegates to i is p_d/n_v . Equation 1 is obtained by taking the products of these terms.

Given the probability of having a partition V^+, V^-, V^i of $V \setminus \{i\}$, the voting power measure for a voter in our PV setting $i \in V$ can be formulated in the following way.

Proposition 2. *Given a bipartite digraph $G = (V, E)$ with $V = (V_d, V_v)$ and $E = \{(i, j) | i \in V_d, j \in V_v\}$ and a binary voting rule W , the LD Penrose-Banzhaf measure $\mathcal{M}_i^{ld}(W, G)$ of voter $i \in V$ can be formulated as:*

$$\begin{aligned} \mathcal{M}_i^{ld}(W, G) &= \sum_{\substack{V_v^+, V_v^- \\ \in \mathcal{P}_2(V_v \setminus \{i\})}} \sum_{\substack{V_d^+, V_d^-, V^i \\ \in \mathcal{P}_3(V_d)}} P_v(n_v^+, n_d^+, n_d^-) \\ &\quad \times \delta_{i, \rightarrow +}^W(V^+, V^-, V^i) \text{ if } i \in V_v. \\ \mathcal{M}_i^{ld}(W, G) &= \sum_{\substack{V_v^+, V_v^- \\ \in \mathcal{P}_2(V_v)}} \sum_{\substack{V_d^+, V_d^- \\ \in \mathcal{P}_2(V_d \setminus \{i\})}} P_d(n_v^+, n_d^+) \\ &\quad \times \delta_{i, \rightarrow +}^W(V^+, V^-, \emptyset) \text{ if } i \in V_d. \end{aligned}$$

We return to Example 1 to illustrate our power measures. Notably, we shall see that a voter in V_v with a small weight can achieve a higher criticality through delegation.

Example 2. *Consider the voters in Example 1; however, now in the PV setting, we assume that $V_v = \{a, b, c\}$ and $V_d = V \setminus V_v$ and we compute the voters' LD Penrose-Banzhaf measures using Proposition 2. The resulting power measures can be seen in Table 1 when $p_d = 0, 0.5, 0.9$. As those in V_d have the possibility of voting directly as well as delegating, they have more influence on the outcome when they are more likely to vote directly; conversely, those in V_v have less as they are less likely to receive delegations. When $p_d = 0$, all agents vote and thus, we return to a standard WVG with the standard Banzhaf measure where all voters with the same weight have the same voting power.*

Computational aspects We turn to some computational aspects regarding the PV setting. We obtain that the exact computation of the LD measure of voting power is #P-hard due to Proposition 1 [Prasad and Kelly, 1990]. More positively, we show that in WVGs, when restricting the underlying graph to represent the PV setting that, \mathcal{M}^{ld} can be computed in pseudo-polynomial time, similarly to the Penrose-Banzhaf measure. This result relies on the following lemma.

Lemma 1. *Given a WVG with weight function w and an integer c . Computing the number of ways of having a partition (S_1, S_2, \dots, S_c) in $\mathcal{P}_c(S)$ of a set $S \subseteq V$*

Table 1: \mathcal{M}^{ld} when $p_d = 0, 0.5, 0.9$ for voters $V = \{a, \dots, m\}$ in the PV setting with $V_v = \{a, b, c\}$ (Values are rounded to 3 d.p.).

Agent $x \in V$	$p_d = 0$	$p_d = 0.5$	$p_d = 0.9$
$a: w = 3$	0.511	0.552	0.542
$b: w = 2$	0.306	0.395	0.438
$c: w = 1$	0.148	0.303	0.390
$d: w = 2$	0.306	0.206	0.138
$V_d \setminus \{d\}: w = 1$	0.148	0.098	0.065

with sizes $n_1, n_2, \dots, n_c = |S| - \sum_{l=1}^{c-1} n_l$, and weights $w(S_1) = w_1, w(S_2) = w_2, \dots$, and $w(S_c) = w_c = w(S) - \sum_{l=1}^{c-1} w_l$ can be computed in pseudo-polynomial time.

Theorem 3. Given a bipartite digraph $G = (V, E)$ with $V = (V_d, V_v)$ and $E = \{(i, j) | i \in V_d, j \in V_v\}$, a WVG with weight function w and quota-ratio q , and a voter i , measure \mathcal{M}_i^{ld} can be computed in pseudo-polynomial time.

5 Liquid democracy with complete digraph

This section discusses the case where $G = (V, E)$ is complete, representing LD where any voter can vote directly, or delegate their vote to any other voter. Since the graph is complete, every voter has the same out-degree $|V| - 1$. Under our individual uniformity assumption, this implies that the probability to delegate p_d is the same for every voter. As with PV, we provide a more compact formulation of our power measure by grouping over similar voters instead of summing over all delegation partitions. By abuse of notation, we say that a set S of voters form an in-forest when the graph obtained by having a vertex per voter in S and an arc from i to j when i delegates to j forms an in-forest. We consider a partition of $V \setminus \{i\}$ into four sets V^+, V^-, V^0, V^i where V^+ (resp. V^-) is a set of n^+ (resp. n^-) voters voting directly in favour of (resp. against) the issue or indirectly by transitively delegating to a root voter in V^+ (resp. V^-); V^0 is a set of n^0 voters abstaining as their delegation leads to a delegation cycle; and V^i is the set of n^i voters delegating (directly or not) to i . Note that V^+, V^-, V^0 and V^i form a partition of $V \setminus \{i\}$.

We will use recursive formulas to compute the probability of having such a partition into four sets. Let $P^{ld}(m, p)$ be the probability that m voters in a set $S \subseteq V$ form an in-forest where the roots all make the same action⁵; an action which is chosen by each root voter with probability p . For instance, $P^{ld}(n^+, p_v/2)$ would be the probability that the voters in V^+ form a forest where each root voter is in favour of the proposal. Consider an arbitrary voter $j \in S$, and a two partition $(S_1, S_2) \in \mathcal{P}_2(S \setminus \{j\})$ with respectively m_1 and $m_2 =$

⁵ $P^{ld}(m, p)$ depends only on $|S|$ and p , and not on the list of voters in S , and thus is independent n the choice of voters in S .

$m - 1 - m_1$ voters. The voters in S_1 are those who delegate directly or indirectly to j , while voters in S_2 do not. Another way of seeing it is that all voters in S_1 form an in-forest where every root delegates to voter j (with probability $p_d/(n-1)$), while voters in S_2 form an in-forest where every root realizes the same action as in S . Regarding voter j , there are two possibilities: either voter j realizes the same action as the roots of S (e.g., voting for the proposal), or they delegate to a member of S_2 (with probability $p_d m_2/(n-1)$).

Hence, we obtain the following recursive formula:

$$P^{ld}(m, p) = \sum_{m_1=0}^{m-1} \binom{m-1}{m_1} P^{ld}(m_1, \frac{p_d}{n-1}) P^{ld}(m_2, p) \times (p + p_d \frac{m_2}{n-1}) \quad (3)$$

with base case $P^{ld}(1, p) = p$ and $P^{ld}(0, p) = 1$.

Thus, the probability that V^+ (resp. V^-) forms an in-forest where the roots vote in favour of (resp. against) the issue is $P^{ld}(n^+, p_v/2)$ (resp. $P^{ld}(n^-, p_v/2)$); and that the probability that V^i forms an in-forest where the roots delegate to voter i is $P^{ld}(n^i, p_d/(n-1))$. For V^0 , we need a different formula. Voters in V^0 have their delegation leading to a delegation cycle through other voters in V^0 iff each voter in V^0 delegates to another voter in V^0 . This occurs with probability $P_0^{ld}(n^0) = (p_d(n^0-1)/(n-1))^{n^0}$.

To sum up, the probability of having a four partition (V^+, V^-, V^i, V^0) of $V \setminus \{i\}$ is $P^{ld}(n^+, p_v/2) P^{ld}(n^-, p_v/2) P^{ld}(n^i, p_d/(n-1)) P_0^{ld}(n^0)$.

Proposition 3. *Given a complete digraph $G = (V, E)$ and a ternary voting rule W , the LD Penrose-Banzhaf measure $\mathcal{M}_i^{ld}(W, G)$ of voter $i \in V$ can be formulated as:*

$$\begin{aligned} \mathcal{M}_i^{ld}(W) &= \sum_{\substack{V^+, V^-, V^0, V^i \\ \in \mathcal{P}_4(V \setminus \{i\})}} P^{ld}(n^+, \frac{p_v}{2}) P^{ld}(n^-, \frac{p_v}{2}) \\ &\times P^{ld}(n^i, \frac{p_d}{n-1}) P_0^{ld}(n^0) \delta_{i, - \rightarrow +}^W(V^+, V^-, V^i). \end{aligned}$$

Example 3. *We return to the agents $V = \{a, \dots, m\}$ from the previous examples, with the same weights as before; however, as we are in the LD setting where the underlying network is a complete digraph. In Table 2, we see the power measures of each agent where the probability of delegating varies. When $p_d = 0$, we are in the standard weighted voting game case where all agents vote directly. When $p_d = 0.5$, those with less voting weight have their voting power measure increase, this is due to the possibility of others delegating to them and the voting weight they control becoming higher. Observe that when $p_d = 1$, all agents are caught in delegation cycles and $T^0 = V$. Thus, we study when $p_d = 0.9$.*

In simulated examples, similar to Example 3, we noticed two trends. First, a *flattening effect* on the power measures as p_d increased. By this, we mean that

Table 2: \mathcal{M}_x^{ld} (rounded to 3 d.p.) for $p_d \in \{0, 0.5, 0.9\}$ for $v = \{a, \dots, m\}$ from Example 1 when considering a complete network.

Agent $x \in V$	$p_d = 0$	$p_d = 0.5$	$p_d = 0.9$
$a: w = 3$	0.511	0.424	0.696
$b, d: w = 2$	0.306	0.308	0.638
$V \setminus \{a, b, d\}: w = 1$	0.148	0.212	0.568

the difference between the lowest and highest measure of power in the WVG (for any agent) becomes smaller. For instance, in Table 2, this difference is 0.438, 0.266, and 0.103 for $p_d = 0, 0.5$, and 0.9 , respectively. This flattening, in our LD setting, is due to all voters having the same available voting actions, no matter their weights. Notably, there can be no dummy agents when $p_d > 0$, as for any agent, the delegation partition where all other voters delegate to them has a positive probability. Second, as illustrated by Table 2, we see that when the probability of delegating increases, so does the probability of being critical, especially when the weights are equal.⁶ As when p_d increases, the number of direct voters decreases while the expected accumulated weight of an agent increases. Hence, they are more likely to be critical when they vote directly. Although it seems intuitive that as the probability of delegating increases, so does the probability of being critical, this is not generally true. In Table 2, we indeed observe that the criticality of voter a decreases when p_d increases from 0 to 0.5.

Computational aspects Using Proposition 3 and Lemma 1, we show that, if the digraph is complete, our power measure can be computed in pseudo-polynomial time for WVGs.

Theorem 4. *Given a complete digraph $G = (V, E)$, a WVG with weight function w and quota-ratio q , and a voter i , \mathcal{M}_i^{ld} can be computed in pseudo-polynomial time.*

The idea of this result is as follows. We can compute the number of ways λ of having a partition (S^1, S^2, S^3, S^4) in $\mathcal{P}_4(V \setminus \{i\})$ with sizes n^+, n^-, n^0, n^i , and weights w^+, w^-, w^0 , and w^i using Lemma 1, and may compute the product $\lambda \times P^{ld}(n^+, \frac{p_d}{2})P^{ld}(n^-, \frac{p_d}{2})P^{ld}(n^i, \frac{p_d}{n-1})P_0^{ld}(n^0)$. The result is the sum of these terms for the different tuples $(n^+, n^-, n^0, n^i, w^+, w^-, w^0, w^i)$ for which i is critical. The number of tuples to be considered is bounded by $n^3 \times w(V)^3$.

6 Experiments

We performed numerical tests on our power measure to test the impact and relationships between different parameters. For each experiment, we estimate

⁶We conjecture that when voting weights are equal, the probability of being critical strictly increases with p_d .

the criticality of voters by sampling over delegation-partitions due to the long runtimes required for exact calculations. Details about the sampling and the confidence interval it entails can be found in Appendix A.5, as well as additional experimental results.

We first computed the criticality of voters with a variety of underlying networks.⁷ First, we observed a strong correlation between the voters’ criticality and their in-degree in the network. This follows the intuition that the higher the in-degree of a voter, the higher the number of voters that can delegate to them. Second, we noticed that the type of the underlying network had a large impact on the differences between the voters’ criticality. In particular, inequality in voting power was the largest on preferential attachment networks [Barabási and Albert, 1999] and the smallest on small-world networks [Watts and Strogatz, 1998].

In the remainder of this section, we focus on our two special cases, bipartite digraphs and complete digraphs.

The number of delegators in proxy voting

In the experiments with proxy voting, we study the case when all voters have the same voting weight and delegators can delegate to any delegatee, as in Section 4. Note that within either V_v or V_d that all voters have the same voting power. We inspect the effect of p_d in the PV setting, i.e., does the probability of those in V_d delegating affect the probability of being critical for both those in V_v and V_d . We set $|V| = 100$ and look at two different number of delegatees, $|V_v| \in \{20, 50\}$.

In Figure 2, when $p_d = 0$, we obtain the standard voting model where all agents vote directly and thus have the same chance of being critical. In both instances, as p_d increases, so does the delegates’ probability of being critical, yet the probability of the delegators being critical decreases, reflecting the intuition that there is some transfer of power from the delegators to the delegates when p_d increases. Observe that the difference between the criticality of the delegators and delegates is smaller when $|V_v| = 50$ than when $|V_v| = 20$ for every value of p_d , as a higher number of delegates share a lower number of delegators. Thus in the PV setting, increasing $|V_v|$ will flatten the probability of being critical.

The effect of the voters’ weights

We study the impact of p_d in the LD model where the underlying network is complete as in Section 5. We have $|V| = 100$ voters, with 50 voters (resp. 30 and 20) having weight 1 (resp. 2 and 5). The quota of the WVG remains $q = 0.5$. We vary p_d between 0 and 0.9. In the case $p_d = 1$, all voters delegate to each other, and thus they all have a criticality of 1. In Figure 3, we see that voters with higher weights have higher voting power. We observe a flattening effect: the initial gap between the criticality of agents with different weights

⁷The specifics of the different networks and the figures for these experiments can be found in Appendix A.5.

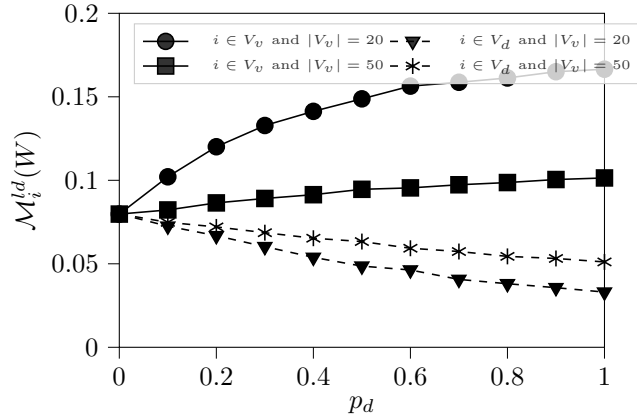


Figure 2: The probability of an agent being critical in the PV setting with p_d varying from 0 to 1. We have $|V| = 100$, $|V_v| \in \{20, 50\}$, and W is a WVG with all weights equal to 1 and $q = 0.5$. This experiment sampled over 100,000 delegations partitions.

gets increasingly smaller when p_d increases. As in Table 2, the criticality of voters with smaller weights always increases while it is not the case for voters with weight 5.

7 Conclusion

This paper continues the tradition of extending the notion of a priori voting power to new voting models. We have introduced the *LD Penrose-Banzhaf measure* to evaluate how critical voters are in deciding the outcome of an election where delegations play a key role. We study a general setting where an underlying graph restricts the possible delegations of the voters. We provided a hardness result on the computation of our measure of voting power. Nevertheless, we designed a sampling procedure to estimate them as well as two pseudo-polynomial algorithms that can be used when the graph restricting the delegations is either bipartite or complete.

Several directions are conceivable for future works. First, one could study the same models with more voting options, such as abstention. We have restricted ourselves to two voting options (approving or disapproving) to keep these new models simple. Another direction would be to find the conditions, such as adding or removing neighbours, that affect the power measure. Additionally, extending the Coleman indices, one could study how to differentiate in our setups the ability to support an initiative from the one to veto it. Lastly, analysing real-election data using our model is a promising option.

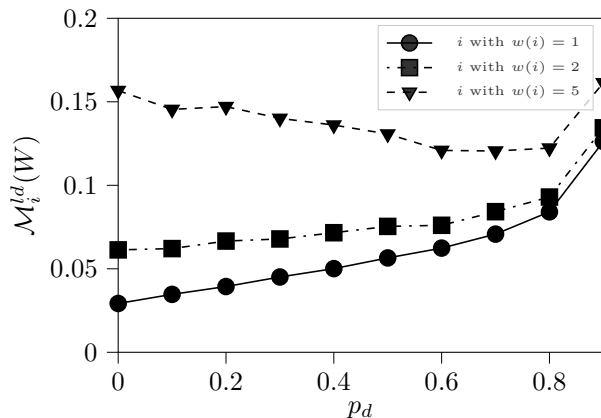


Figure 3: Probability of an agent being critical with p_d varying from 0 to 0.9 when the underlying graph is complete. We have $V = 100$, and W is a WVG with 50 (resp. 30, 20) voters with weights equal to 1 (resp. 2, 5) and $q = 0.5$. This experiment sampled over 10,000 delegations partitions.

Acknowledgments

Rachael Colley acknowledges the support of the ANR JCJC project SCONE (ANR 18-CE23-0009-01). Théo Delemazure was supported by the PRAIRIE 3IA Institute under grant ANR-19-P3IA-0001 (e). Hugo Gilbert acknowledges the support from the project THEMIS ANR-20-CE23-0018 of the French National Research Agency (ANR).

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A Appendix of Paper

A.1 Omitted Proofs

Theorem 1. *Given a digraph $G = (V, E)$, a ternary voting rule W , and a voter $i \in V$, we have that:*

$$\mathbb{P}(i \text{ is critical}) = \mathcal{M}_i^{ld}(W, G).$$

Moreover, if $\text{NB}_{out}(i) = \emptyset$ or W is symmetrical, we have that:

$$\begin{aligned} \mathbb{P}(i \text{ is positively critical}) &= \mathcal{M}_i^{ld}(W, G)/2 \\ &= \mathbb{P}(i \text{ is negatively critical}). \end{aligned}$$

Proof sketch. $\mathcal{M}_i^{ld}(W, G) = \mathbb{P}(i \text{ is critical})$ as it sums the probability of a delegation-partition being critical, determined by $W(T_{D_i^+}) - W(T_{D_i^-})$. Being positively critical means that changing a vote for to against the issue will also change the outcome in the same way (negatively critical is defined similarly). If W is symmetric then $W(T_{D_i^+}) - W(T_{D_i^-})$ captures both i being positively or negatively critical. If $\text{NB}_{out}(i) = \emptyset$, then their only option is to vote either for or against the issue. In both cases, $\mathbb{P}(i \text{ is positively critical}) = \mathcal{M}_i^{\gamma}(W)/2 = \mathbb{P}(i \text{ is negatively critical})$. \square

Theorem 2. *Given a digraph $G = (V, E)$ and a WVG defined on V , computing the LD-Penrose-Banzhaf power measure of a voter is $\#P$ -hard under Turing reductions even when voter's weights are bounded linearly by the number of voters.*

Proof. To prove this, we will give a reduction from the problem of counting simple paths in a digraph which is known to be $\#P$ -complete [Valiant, 1979]. The problem takes as input a directed graph $G = (V, E)$ and nodes $s, t \in V$, the problem then returns the number of simple paths from s to t in G .

Let $G = (V, E)$ be a directed graph and $s, t \in V$ two connected vertices of this graph. We want to count simple paths starting at s and finishing at t . In order for our problems to align, we will make some alterations to G . We remove every incoming edge of s and every outgoing edge of t from E and we add a dummy vertex z to V with no outgoing or incoming edges.

Let d_{max} be the highest out-degree of any vertex in G . Let $k \geq 0$ and $G_k = (V_k, E_k)$ be a graph obtained from G , such that $V_k = V \cup \{z\} \cup D_k$ with $|D_k| = d_{max} + k$ a set of dummy vertices. For every vertex $x \in V \setminus \{t, z\}$, we add edges from x to $d_{max} + k - \text{NB}_{out}(x)$ dummy vertices from D_k to get E_k . By doing this, we ensure that every vertex in V_k either have $d_{max} + k$ or 0 out-neighbours. Note that we have only added a polynomial number of vertices and edges to G to obtain G_k , if k is smaller than some polynomial in n .

Note that the maximum length of a simple path from s to t in G is less than the total number n of vertices in G . Now, we will compute the criticality of t in the graph G_k with $k = 0, \dots, n$ for the WVG in which $w(s) = 3|V_k|$, $w(z) = |V_k|$ and $w(x) = 1$ for all $x \in V_k \setminus \{s, z\}$ and $q = \frac{1}{2}$.

Observe that when s is not in a cycle, then s is in some sense a dictator as $w(s) > q \sum_{x \in V_k} w(x) = \frac{1}{2}(5|V_k| - 2)$ and the election's outcome is determined by their vote. However, if s is in a cycle, then the outcome of the election depends only on z 's vote, $w(z) > q \sum_{x \in V_k \setminus \{s\}} w(x) = \frac{1}{2}(2|V_k| - 2)$ (furthermore, note that z cannot be in a cycle as they have no outgoing edges). Thus, a vertex $x \in V_k \setminus \{s, z\}$ is critical given a delegation partition iff x is part of s 's delegation path. In particular, this is true for node t .

Let P be a delegation path from s to t in G_k . Each vertex on this path has $d_{max} + k$ outgoing edge by definition (otherwise they would have 0 outgoing edges and could not be on this path). Thus, the probability of obtaining this path of length ℓ is equal to $\left(p_d^{d_{max} + k} \frac{1}{d_{max} + k}\right)^\ell$ under the individual uniformity assumption, where the probability to delegate $p_d^{d_{max} + k}$ is the same for all voters as they all have the same number of outneighbors. We define $\pi_k = p_d^{d_{max} + k} \frac{1}{d_{max} + k}$ and let \mathcal{P}_ℓ be the set of paths of length ℓ between s and t in G , for $\ell \leq n$. We know that t is critical iff there is a path from s to t on the delegation graph, and that there cannot be two different paths from s to t in the same delegation graph. Thus, the criticality of t is equal to the sum of the probabilities of every path from s to t to be selected in the delegation graph:

$$\mathbb{P}(t \text{ is critical in } G_k) = \sum_{\ell=0}^n |\mathcal{P}_\ell| \pi_k^\ell. \quad (4)$$

Thus, from Equation 4 we obtain $n + 1$ linear equations (one from each election, i.e., each value of k) where the variables are $|\mathcal{P}_\ell|$. This gives us a set of $n + 1$ equations, which is polynomial in the number of vertices.

The coefficient matrix M of these equations is a Vandermonde matrix with $M_{ij} = \pi_i^j$ for $i, j = 0, \dots, n$ and all the π_i values are different. Thus, if we denote the two vectors $X^P = (|P_0|, \dots, |P_n|)$ and Y^c such that $Y^c = \mathbb{P}(t \text{ is critical in } G_k)$, we have the equation $Y^c = M X^P$. As M is a Vandermonde matrix with different coefficients, it is easily invertible and X^P could be computed in polynomial time. Hence, we would obtain the values of $|P_1|, \dots, |P_n|$ and we could easily derive the total number of paths from s to t in G by doing $\sum_{\ell=0}^n |P_\ell|$. This would solve the #P-complete problem of counting simple paths from s to t . \square

Lemma 1. *Given a WVG with weight function w and an integer c . Computing the number of ways of having a partition (S_1, S_2, \dots, S_c) in $\mathcal{P}_c(S)$ of a set $S \subseteq V$ with sizes $n_1, n_2, \dots, n_c = |S| - \sum_{l=1}^{c-1} n_l$, and weights $w(S_1) = w_1, w(S_2) = w_2, \dots$, and $w(S_c) = w_c = w(S) - \sum_{l=1}^{c-1} w_l$ can be computed in pseudo-polynomial time.*

Proof. We first order (arbitrarily) the voters in S such that v_i is the i^{th} voter in S . We will denote by $S[i]$ the subset $\{v_i, v_{i+1}, \dots, v_{|S|}\}$ of S . The proof then

relies on the following recursive formula:

$$\Lambda[i, n_1, \dots, n_{c-1}, w_1, \dots, w_{c-1}] = \Lambda[i+1, n_1-1, n_2, \dots, n_{c-1}, w_1-w(v_i), w_2, \dots, w_{c-1}] \quad (5)$$

$$+ \Lambda[i+1, n_1, n_2-1, \dots, n_{c-1}, w_1, w_2-w(v_i), \dots, w_{c-1}] \quad (6)$$

\vdots

$$+ \Lambda[i+1, n_1, n_2, \dots, n_{c-1}-1, w_1, w_2, \dots, w_{c-1}-w(v_i)]$$

$$+ \Lambda[i+1, n_1, n_2, \dots, n_{c-1}, w_1, w_2, \dots, w_{c-1}]. \quad (7)$$

The term $\Lambda[i, n_1, n_2, \dots, n_{c-1}, w_1, w_2, \dots, w_{c-1}]$ denotes the number of ways of having a group of n_1 voters in $S[i]$ with total weight w_1 , and n_2 other voters in $S[i]$ having total weight w_2, \dots , and n_{c-1} other voters in $S[i]$ having total weight w_{c-1} . The number of ways of having a partition (S_1, S_2, \dots, S_c) in $\mathcal{P}_c(S)$ with sizes $n_1, n_2, \dots, n_c = |S| - \sum_{l=1}^{c-1} n_l$, and weights $w(S_1) = w_1, w(S_2) = w_2, \dots$, and $w(S_c) = w_c = w(S) - \sum_{l=1}^{c-1} w_l$ is then $\Lambda[1, n_1, n_2, \dots, n_{c-1}, w_1, w_2, \dots, w_{c-1}]$. Let us now explain the recursive formula: the term on line 5 (resp. line 6, line 7) counts the number of such partitions of S when v_i is part of the first group of n_1 voters (resp. second group of n_2 voters, neither of the $c-1$ first groups, hence the last group). The base cases are as follows: $\Lambda[i, n_1, \dots, n_{c-1}, w_1, \dots, w_{c-1}] = 0$ if at least one of the parameters is inferior to 0; $\Lambda[i, 0, 0, \dots, 0] = 1$ for $i \in \{1, \dots, |S| + 1\}$; and $\Lambda[|S| + 1, n_1, \dots, n_{c-1}, w_1, \dots, w_{c-1}] = 0$ if at least one of the $2(c-1)$ last parameters is different from 0. For a fixed c value, using memoization, this recursive formula can be computed in polynomial time with respect to n and $\max_v w(v)$. \square

Theorem 3. *Given a bipartite digraph $G = (V, E)$ with $V = (V_d, V_v)$ and $E = \{(i, j) | i \in V_d, j \in V_v\}$, a WVG with weight function w and quota-ratio q , and a voter i , measure \mathcal{M}_i^{ld} can be computed in pseudo-polynomial time.*

Proof. We give the details of the more complex case when $i \in V_v$. We consider all the possibilities of having a two-partition of $V_v \setminus \{i\}$ with sets of sizes n_v^+ and n_v^- and weights w_v^+ , and w_v^- in conjunction with a three partition of V_d with sets of sizes n_d^+, n_d^-, n^i , and weights w_d^+, w_d^-, w^i . Such a tuple (with 10 elements) will be called a *decomposition* of V informally. The number of such decompositions is bounded by $n_v \times n_d^2 \times w(V)^3$. Given a decomposition, we say that i is critical if $w_v^+ + w_d^+$ is in the interval $(q \times w(V) - w(i) - w^i, q \times w(V)]$.

On the one hand, we compute the number of ways λ_1 of having a partition (S^1, S^2, S^3) in $\mathcal{P}_3(V_d)$ with sizes n_d^+, n_d^-, n^i , and weights w_d^+, w_d^-, w^i . On the other hand, we compute the number of ways λ_2 of having a partition (S^1, S^2) in $\mathcal{P}_2(V_v \setminus \{i\})$ with sizes n_v^+, n_v^- , and weights w_v^+, w_v^- . Both operations are performed using Lemma 1 in pseudo-polynomial time. Last, we compute the product $\lambda_1 \times \lambda_2 \times P_v(n_v^+, n_d^+, n_d^-)$. The result is the sum of these terms for the different possible decompositions for which i is critical. \square

Theorem 4. *Given a complete digraph $G = (V, E)$, a WVG with weight function w and quota-ratio q , and a voter i , \mathcal{M}_i^{ld} can be computed in pseudo-polynomial time.*

Proof. Let $w_{\max} = \max_v w(v)$. We consider all the possibilities of having a four partition of $V \setminus \{i\}$ with sets of sizes n^+ , n^- , n^0 , and n^i and weights w^+ , w^- , w^0 , w^i . Such a tuple (with 8 elements) will be called a *decomposition* of V informally. The number of such decompositions is bounded by $n^3 \times w(V)^3$. Given a decomposition, we say that i is critical if w^+ is in $(q \times (w(V) - w^0) - w(i) - w^i, q \times (w(V) - w^0)]$.

Then, we compute the number of ways λ of having a partition (S^1, S^2, S^3, S^4) in $\mathcal{P}_4(V \setminus \{i\})$ with sizes n^+ , n^- , n^0 , n^i , and weights w^+ , w^- , w^0 , and w^i using Lemma 1. Last, we compute the product $\lambda \times P^{ld}(n^+, \frac{p_v}{2}) P^{ld}(n^-, \frac{p_v}{2}) P^{ld}(n^i, \frac{p_d}{n-1}) P_0^{ld}(n^0)$. The result is the sum of these terms for the different possible decompositions for which i is critical. \square

A.2 The Delegative Banzhaf measure as a measure of I-power.

To differentiate our measures from the delegative Banzhaf measure defined by Zhang and Grossi 2021, we now provide a probabilistic model on voters' behaviours to rationalize it as a measure of I-power.

Similarly, as in WVGs, the authors study elections in which each voter has a voting weight, and a coalition wins if and only if its "accumulated weight" exceeds the quota. Additionally, we are given an in-forest where vertices are voters and an arc from i to j represents that i delegates to j . Thus, each voter in an in-forest has a direct voter (the corresponding root voter) via a chain of delegations who will vote on their behalf. Unlike in standard WVGs, a coalition's accumulated weight is not considered as the sum of its members' weights but only those for which the delegation chain to their direct voter is included in the coalition. Hence, a voter whose chain of delegation requires a voter outside of the coalition to reach their root voter will not contribute to the weight of the coalition. The delegative Banzhaf measure is then defined as the standard Banzhaf-Penrose measure in this voting game.

This measure can be rationalized by the following probabilistic model on voters' behaviours. Each voter has a probability 0.5 of having a positive a priori opinion about the proposal and a probability 0.5 of having a negative opinion about the proposal (these probabilities being independent); this leads to a uniform probability distribution on bipartitions of the voters set. Given a bipartition of opinions and an in-forest, we define a direct-voting partition in the following recursive way: each voter votes in favour of the proposal if i) they have a positive opinion about the proposal, and ii) either they vote directly or the person they directly delegated to in the in-forest also votes in favour of the proposal. If one of these conditions is not met, the voter will vote against the proposal. Put differently, the in-forest provided in input gives each voter a condition for their support. The voting rule used is then the same as in standard

WVGs. Given this probabilistic model, the delegative Banzhaf measure is the probability of being critical.

As one can see, the delegative Banzhaf measure differs from the power measures we define as the input to define them and the intuitions underlying them both differ.

A.3 Connection between WVGs with randomised weights and our proxy voting settings

Weighted voting games in our proxy setting, i.e., when the underlying network is a bipartite digraph, can be seen as a special case of weighted voting games with random weights where the weights are obtained through our probabilistic model on delegations.

However, in this section, we will stress some differences between the settings. First, note that the set of voters voting directly is also randomised as members of V_d may or may not vote. Moreover, the random processes generating the weights for the delegates are specific to our proxy settings. These weights are constrained by the weights of the delegators and our probabilistic delegation model. Boratyn *et al.* 2020 claim to be the first to study the Banzhaf index with random weights. In their setting, the voters' weights are drawn uniformly from a probability simplex, providing a continuous distribution from which to draw their (normalised) weights. This entails that the discrete nature of the delegators' weights cannot be adequately modeled within this framework as there is no way to ensure that the weights drawn correspond to an actual coalition (for example, that some delegator's weight isn't being split across two delegates). Instead of using the uniform distribution, one could try to use the probability distribution on the weight simplex yielded by our proxy voting models. Unfortunately, note that this probability distribution would have a support that at worst is of exponential size. Hence, it seems difficult to express this probability distribution compactly while ignoring the delegation process that generates it.

In other papers that study the Shapley-Shubik index [Filmus *et al.*, 2016; Bachrach *et al.*, 2016], the authors consider weighted-voting games where voters' weights are drawn independently from one another under some probability distribution. Once again, because of this independence assumption, these models do not seem to encapsulate our proxy voting settings.

A.4 An alternative proxy voting model

In this section, we explore an alternative proxy voting model in which V is partitioned in two sets V_v of size n_v and V_d of size $n_d = n - n_v$. Similarly to the PV model discussed in our paper, the set V_v contains delegates who will vote. Differently, voters in V_d can only delegate their votes to proxies in V_v . Hence, this assumption restricts the class of admissible delegation partitions. We term this restricted setting of proxy voting PV_r .

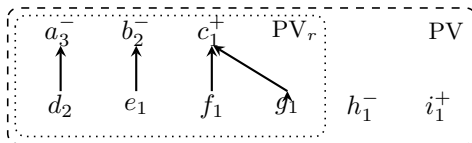


Figure 4: This figure depicts the agents' delegations in Example 4 such that the subscript gives the agent's weight and the superscript of $+$ or $-$ represents the direct vote of the non-delegating agents. When restricting the agents to be those inside the dotted, and dashed lines, we obtain a valid PV_r , and PV-partition, respectively.

Definition 5. Given a non-empty set $V_v \subseteq V$, a PV_r -partition P is a map D who takes value in V and for which $D(v) \in \{-1, 1\}$ if $v \in V_v$, and $D(v) \in V_v$ otherwise.

We now provide an example to illustrate the differences between our two PV settings.

Example 4. Consider a set of agents $V = \{a, b, \dots, i\}$ with $V_v = \{a, b, c\}$ such that $D^+ = \{c, i\}$, $D^- = \{a, b, h\}$, $D^a = \{d\}$, $D^b = \{e\}$, and $D^c = \{f, g\}$ as described in Figure 4. This delegation partition is valid for the PV model described in our paper where $G = (V, E)$ is a bipartite graph, in which $V = (V_d, V_v)$ and $E = \{(i, j) | i \in V_d, i \in V_v\}$. However, while this example, restricted to voters $\{a, b, \dots, g\}$ form a valid PV_r -partition, it is not the case when considering voters h and i as they vote directly and yet belong to V_d .

We now revisit the individual uniformity and global uniformity assumptions under the lens of this model.

The individual uniformity assumption Under the individual uniformity assumptions, each voter in V_v has equal probability of supporting or opposing the proposal. Additionally, each voter in V_d has equal probability of delegating to any voter in V_v . Voters behaviors are independent from one another. These assumptions are justified by the fact that we have no information as to voters' personality and interests, or the nature of the proposal and we are ignorant of any concurrence or opposition of interests between voters. Hence, we assume that $p_y = p_n = 1/2$ for voters in V_v and that voters in V_d may delegate to any member of V_v with probability $1/n_v$.

The global uniformity assumption If there is no information about the proposal or voters, we assume that all PV_r partitions are equally likely. We call this assumption the *global uniformity assumption*, as it does not rely directly on a model of individual behaviours. Hence, the probability of a given partition being chosen is $(1/2)^{n_v} \times (1/n_v)^{n_d}$.

Remark. Note that the *individual* and *global* uniformity assumptions lead to the same probabilistic model under the the PV_r setting.

A measure of voting power for the PV_r setting We want to measure how critical a voter is in determining the outcome. Given our probabilistic model on PV_r partitions, we consider the number of times a voter can change the outcome decided by a binary voting rule.

- If a voter belongs to V_v , then we look if the agent can impact the outcome by changing their vote, either to be against the proposal (positive criticality) or in favour of the proposal (negative criticality).
- If a voter belongs to V_d , then we look if an agent has impact by changing their delegation (if possible), either by switching from a delegatee in favour to one against the proposal (positive criticality) or from a delegatee against to one in favour of the proposal (negative criticality).

Given a PV_r -partition, the agent is critical if they are positively or negatively critical.

To measure how critical an agent i is, we consider a partition of $V \setminus \{i\}$ into three sets V^+ , V^- , V^i :

- V^+ represents the n^+ voters whose final vote is in favour of the proposal, either by voting directly or indirectly by delegating to a delegatee in $V_v \setminus \{i\}$.
- V^- represents the n^- voters who vote directly against the proposal or indirectly by delegating to a delegatee in $V_v \setminus \{i\}$.
- V^i is the set of n^i voters who delegate to voter i . This set is empty if $i \in V_d$.

Note that $V^+ \cup V^- \cup V^i = V \setminus \{i\}$ and that these partitioning sets are disjoint. We will focus on how these sets intersect V_d and V_v . We define V_d^+ , V_d^- , V_v^+ , and V_v^- with size n_d^+ , n_d^- , n_v^+ , and n_v^- , respectively, such that $V_x^\circ = V_x \cap V^\circ$ for $x \in \{v, d\}$ and $\circ \in \{-, +\}$.

Given our probabilistic model on proxy partitions, we observe that the probability of having a partition V^+ , V^- , V^i only depends on these cardinality values.

- If $i \in V_v$, it only depends on n_v^+ , n_d^+ , and n_d^- , as $n_v^- = n_v - n_v^+ - 1$ and $n^i = n_d - n_d^+ - n_d^-$. Hence, we then denote this probability by $P_v^r(n_v^+, n_d^+, n_d^-)$.
- If $i \in V_d$, it only depends on n_v^+ , and n_d^+ as $n_v^+ + n_v^- = n_v$ and $n_d^- = n_d - 1 - n_d^+$. Hence, we then denote this probability by $P_d^r(n_v^+, n_d^+)$.

The formulas to compute these probabilities are given below:

$$P_v^r(n_v^+, n_d^+, n_d^-) = \frac{(n_v^+)^{n_d^+} (n_v^-)^{n_d^-}}{2^{n_v-1} n_v^{n_d}}, \quad (8)$$

$$P_d^r(n_v^+, n_d^+) = \frac{(n_v^+)^{n_d^+} (n_v^-)^{n_d^-}}{2^{n_v} n_v^{n_d-1}}. \quad (9)$$

Note that there are some obvious conditions on the integer parameters n_v^+ , n_d^+ , and n_d^- .

- If $i \in V_v$, it should be that $n_v^+ \leq n_v - 1$, and $n_d^+ + n_d^- \leq n_d$. Moreover, values n_d^+ (resp. n_d^-) should be equal to 0 if $n_v^+ = 0$ (resp. $n_v^+ = n_v - 1$).
- If $i \in V_d$, we should now have $n_v^+ \leq n_v$, and $n_d^+ + n_d^- = n_d - 1$. Once more, values n_d^+ (resp. n_d^-) should be equal to 0 if $n_v^+ = 0$ (resp. $n_v^+ = n_v$).

If some of these conditions are not respected, we set that $P_v^r(n_v^+, n_d^+, n_d^-) = 0$ or $P_d^r(n_v^+, n_d^+) = 0$. Let us now detail Equation 8 (Equation 9 is obtained in a similar way). The probability of the binary votes of the delegates other than i being a certain way is $(1/2)^{n_v-1}$. Then, the probability that voters in V_d^+ (resp. V_d^- , V^i) delegate to the ones in V_v^+ (resp. V_v^- , $\{i\}$) is $(n_v^+/n_v)^{n_d^+}$ (resp. $(n_v^-/n_v)^{n_d^-}$, $(1/n_v)^{n_d^i}$). Equation 8 is obtained by considering the products of these terms.

Given the probability of having a partition V^+ , V^- , V^i of $V \setminus \{i\}$, the voting power measure for a voter $i \in V$ can be defined.

Definition 6. Given a set V of voters, a set $V_v \subseteq V$ of delegates, and a binary voting rule W , the PV_r Penrose-Banzhaf measure $\mathcal{M}_i^r(W)$ of voter $i \in V$ is defined as:

$$\begin{aligned} \mathcal{M}_i^r(W) &= \sum_{\substack{V_v^+, V_v^- \\ \in \mathcal{P}_2(V_v \setminus \{i\})}} \sum_{\substack{V_d^+, V_d^-, V^i \\ \in \mathcal{P}_3(V_d)}} P_v^r(n_v^+, n_d^+, n_d^-) \\ &\quad \delta_{i, \rightarrow \rightarrow +}^W(V^+, V^-, V^i) \text{ if } i \in V_v, \\ \mathcal{M}_i^r(W) &= \sum_{\substack{V_v^+ \neq \emptyset, V_v^- \neq \emptyset \\ \in \mathcal{P}_2(V_v)}} \sum_{\substack{V_d^+, V_d^- \\ \in \mathcal{P}_2(V_d \setminus \{i\})}} P_d^r(n_v^+, n_d^+) \\ &\quad \delta_{i, \rightarrow \rightarrow +}^W(V^+, V^-, \emptyset) \text{ if } i \in V_d. \end{aligned}$$

Note that when $i \in V_d$, we consider only partitions for which $V_v^+ \neq \emptyset$ and $V_v^- \neq \emptyset$. Indeed, those in V_d are only able to delegate to delegates. Hence, their ability to be critical depends on the fact that two delegates with opposite votes exist. Note that the probability that all delegates vote in the same way is $(1/2)^{n_v-1}$.

\mathcal{M}_i^r quantifies the probability to sample a PV_r -partition where i is able to alter the election's outcome. Additionally, observe that our voting power

measures extend the standard Penrose-Banzhaf measure (consider $V_v = V$) and that they are not normalized (i.e., summing over the agents does not yield 1). The corresponding voting power indices can be found by normalizing over voters.

Computational aspects We now turn to some computational aspects regarding the PV_r measure of voting power. While the exact computation of this measure is #P-hard as it extends the standard Banzhaf measure, we show that in WVGs, measures \mathcal{M}^r can be computed in pseudo-polynomial time. This result relies on Lemma 1.

Theorem 5. *Given a WVG with weight function w and quota-ratio q , a set of voters $V_v \subseteq V$, and a voter i , measure \mathcal{M}_i^r can be computed in pseudo-polynomial time.*

Proof. We give the details of the more complex case when $i \in V_v$. We consider all the possibilities of having a two partition of $V_v \setminus \{i\}$ with sets of sizes n_v^+ and n_v^- and weights w_v^+ , and w_v^- in conjunction with a three partition of V_d with sets of sizes n_d^+ , n_d^- , n^i , and weights w_d^+ , w_d^- , and w^i . Such a tuple (with 10 elements) will be called a *decomposition* of V informally. The number of such decompositions is bounded by $n_v \times n_d^2 \times w(V)^3$. Given a decomposition, we say that i is critical if $w_v^+ + w_d^+$ is in the interval $(q \times w(V) - w(i) - w^i, q \times w(V)]$.

On the one hand, we compute the number of ways λ_1 of having a partition (S^1, S^2, S^3) in $\mathcal{P}_3(V_d)$ with sizes n_d^+ , n_d^- , n^i , and weights w_d^+ , w_d^- , and w^i . On the other hand, we compute the number of ways λ_2 of having a partition (S^1, S^2) in $\mathcal{P}_2(V_v \setminus \{i\})$ with sizes n_v^+ , n_v^- , and weights w_v^+ , and w_v^- . Both operations are performed using Lemma 1 in pseudo-polynomial time. Last, we compute the product $\lambda_1 \times \lambda_2 \times P_v^r(n_v^+, n_d^+, n_d^-)$. The result is the sum of these terms for the different possible decompositions for which i is critical. \square

A.5 Additional details on the experiments

Sampling procedure. We show that we can estimate the values $\mathcal{M}_i^{ld}(W, G)$ by a sampling procedure which samples “enough” G -delegation partitions according to the individual uniformity assumption and counting the number of samples for which i is critical. To determine how many G -delegation partitions should be sampled, we rely on the following well known inequality [Hoeffding, 1994].

Theorem 6 (Hoeffding’s inequality). *Let X_1, \dots, X_m be independent random variables, where all X_i are bounded such that $X_i \in [a_i, b_i]$, and let $X = \sum_{i=1}^m X_i$. Then the following inequality holds.*

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq n\epsilon) \leq 2 \exp\left(-\frac{2m^2\epsilon^2}{\sum_{i=1}^m (b_i - a_i)^2}\right)$$

Using this inequality, we can prove the following result.

Theorem 7 (Estimation by sampling). *Given a graph $G = (V, E)$, a ternary voting rule W , values $\epsilon > 0$, and $\delta > 0$, let $\tilde{\mathcal{D}}$ be a set of $k \geq \ln(2n/\delta)/(2\epsilon^2)$ G -delegation partitions sampled independently and uniformly. Then,*

$$\tilde{\mathcal{M}}_i^{ld}(W, G) = \frac{1}{k} \sum_{D \in \tilde{\mathcal{D}}} \frac{W(T_{D_i^+}) - W(T_{D_i^-})}{2}.$$

belongs to $[\mathcal{M}_i^{ld}(W, G) - \epsilon, \mathcal{M}_i^{ld}(W, G) + \epsilon]$ with probability $1 - \delta/n$. By using a union bound, the result holds for all voters with probability $1 - \delta$.

Proof. Note that $\frac{W(T_{D_i^+}) - W(T_{D_i^-})}{2}$ is a random variable taking value in $\{0, 1\}$. Hence, using Hoeffding's inequality, the fact that $k \geq \ln(2n/\delta)/(2\epsilon^2)$ and that $\mathcal{M}_i^{ld}(W, G) = \mathbb{E}\left(\frac{W(T_{D_i^+}) - W(T_{D_i^-})}{2}\right)$, we obtain that for any $i \in V$,

$$\mathbb{P}(|\mathcal{M}_i^{ld}(W, G) - \tilde{\mathcal{M}}_i^{ld}(W, G)| \geq \epsilon) \leq \delta/n.$$

For the next step, we use a union bound.

Proposition 4 (Union bound). *Let E_1, \dots, E_m be m events, then: $\mathbb{P}(\bigcup_{i=1}^m E_i) \leq \sum_{i=1}^m \mathbb{P}(E_i)$.*

Hence,

$$\begin{aligned} \mathbb{P}(\exists i \in V \text{ s.t. } |\mathcal{M}_i^{ld}(W, G) - \tilde{\mathcal{M}}_i^{ld}(W, G)| \geq \epsilon) &\leq \\ \sum_{i \in V} \mathbb{P}(|\mathcal{M}_i^{ld}(W, G) - \tilde{\mathcal{M}}_i^{ld}(W, G)| \geq \epsilon) &\leq \delta \end{aligned}$$

□

We first ran experiments to compute the criticality of voters in different types of network, to see if we could find correlations between the criticality and other metrics.

The kind of networks we considered for this first experiments were the following:

- **Random graph** $G(n, p)$ with n nodes and each edge having a probability p to be added to the network.
- **Preferential attachment model** based on the model from [Barabási and Albert, 1999] parameterized by the number of nodes n and the number of edges m to add for each new node. For a network of n nodes, we sequentially add the nodes to the graph and when we add node we also attach it to m of the existing nodes, with a probability proportional to their current degree in the graph. More formally, let V be the set of existing nodes when we add a new node $x \notin V$, then the probability to add the edge (x, v) for some $v \in V$ is proportional to the current degree of v . This follows the “rich get richer” idea as nodes that are already important in the network are more likely to become even more important.

- **Small world model** based on the model from [Watts and Strogatz, 1998]. On the contrary to the previous model, the idea is here that no nodes accumulate too many connections, by mainly having edges with its neighbourhood. More formally, for a network of n nodes, we associate each node to a position on a ring and we connect every node to its k nearest neighbours, and every edge is rewired with some probability p , given as a parameter. If an edge is rewired for one node, it can be rewired to any other node of the network uniformly at random. In our experiments, we used $p = 0.2$

Note that in our experiments, we considered the versions of these three models that gave undirected graphs, so every edge is bidirectional. The next two models give us directed graphs.

- In the **spatial model**, every node is first embedded in a 2 dimensional Euclidean space according to some distribution of positions \mathcal{D} , and for each node, we add a directed edge to its k nearest neighbours on this Euclidean space. Note that this graph is directed as one voter node be another node nearest neighbours while the converse is false. We considered two distribution of positions: (i) the Uniform distribution in $[-1, 1]^2$ and the Gaussian distribution centered in 0 with standard deviation 1. We expect that with the Gaussian distribution, nodes that are embedded close to the center of the plane $(0, 0)$ get an advantage against the nodes that have less central positions.
- **k -layers model** are graph with k layers, each layer having the same number of nodes, and every node from the layer $j < k$ has a directed edge towards every node from the layer $j + 1$.

We constructed 5 networks of each family with $n = 100$ voters and set up the parameters m , k or p such that the average in-degree of the graph is 10. For each network, we ran 5,000 delegations-partitions. The figures are averaged over the 5 networks for each type of networks.

Figure 5 shows the distribution of criticality for every type of graph, by ordering the voters from the most critical to the least critical. We observe that for some networks, there are voters that are far more critical than others. This is the case for the preferential attachment model in which the most critical voter has $\mathcal{M}_i^{ld}(W) = 0.35$ and the least critical voter has 0.063. Whereas, voters in networks based on the small world model seem to have almost all the same criticality, as the most critical voter has $\mathcal{M}_i^{ld}(W) = 0.159$ and the least critical voter has 0.094. We note that in spatial networks, the differences of criticality are greater with the Gaussian distribution than with uniform distribution. In particular, we observe that some voters seems to have very little power in deciding the outcome of the vote. Finally, for the k -layer experiment (with $k = 10$), it is clear that voters from each layer have (almost) the same criticality and that the criticality gets higher with the layer. This can also be seen in Figure 7.b. In particular, voters on the last layer (with no outgoing edges, so they have

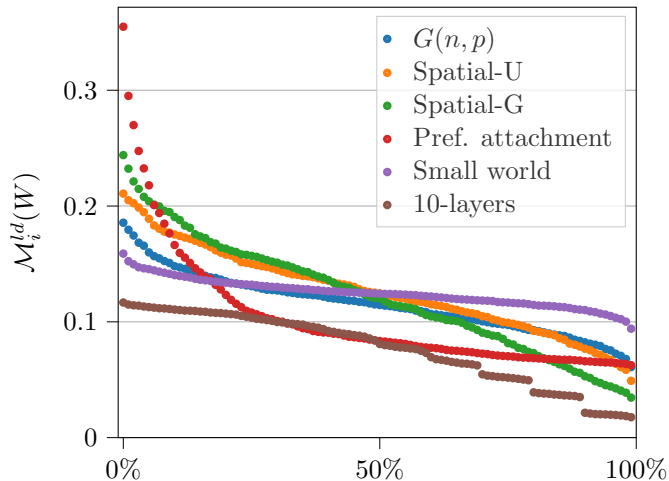


Figure 5: Distribution of the criticality among the voters in the network, from the most critical to the least critical (thus the percentage on the x-axis represents the percentage of voters considered).

to vote), are the one with the highest criticality in average. However, one can note that if the criticality increases quickly between the first layers, it will then increase more slowly between the last ones.

We also looked at other metrics of the networks we generated. In particular, we noticed that the criticalities seemed to be very correlated to the degree. Figure 6 shows the distribution of in-degrees in the graph, from the node with the highest in-degree, to the node with the lowest one. We observe that the curves of this figures seems very similar to the ones from Figure 5.

To observe more clearly the correlations between the in-degree and the criticality, we plotted for each kind of network, the nodes by their in-degree and their criticality. The results are shown in Figure 7. We see the large correlation between the two metrics, especially in the non-directed networks ($G(n, p)$, preferential attachment and small world). However, in the spatial graph, we see some variations between nodes having the same in-degree. This makes sense, as in these graph the positions of the in-neighbours is maybe as important as their number. In the k -layer model, every node has in-degree either 10 or 0 (for the first layer), see instead of the degree, we look at how critical the nodes of each layer are.

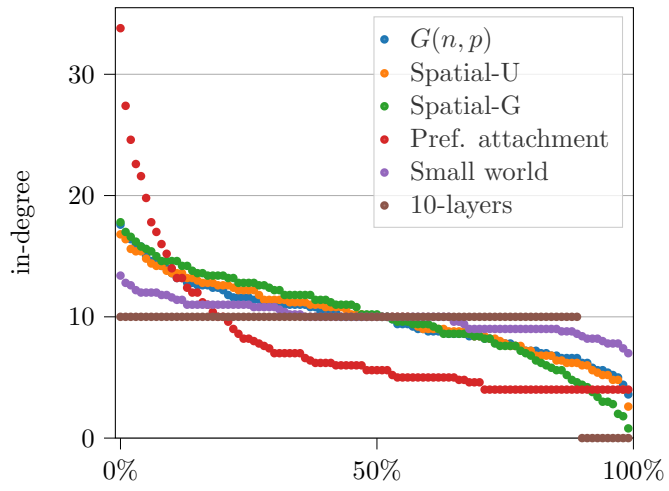


Figure 6: Distribution of the in-degrees among the voters in the network, from the voter with the highest in-degree to the one with the lowest in-degree.

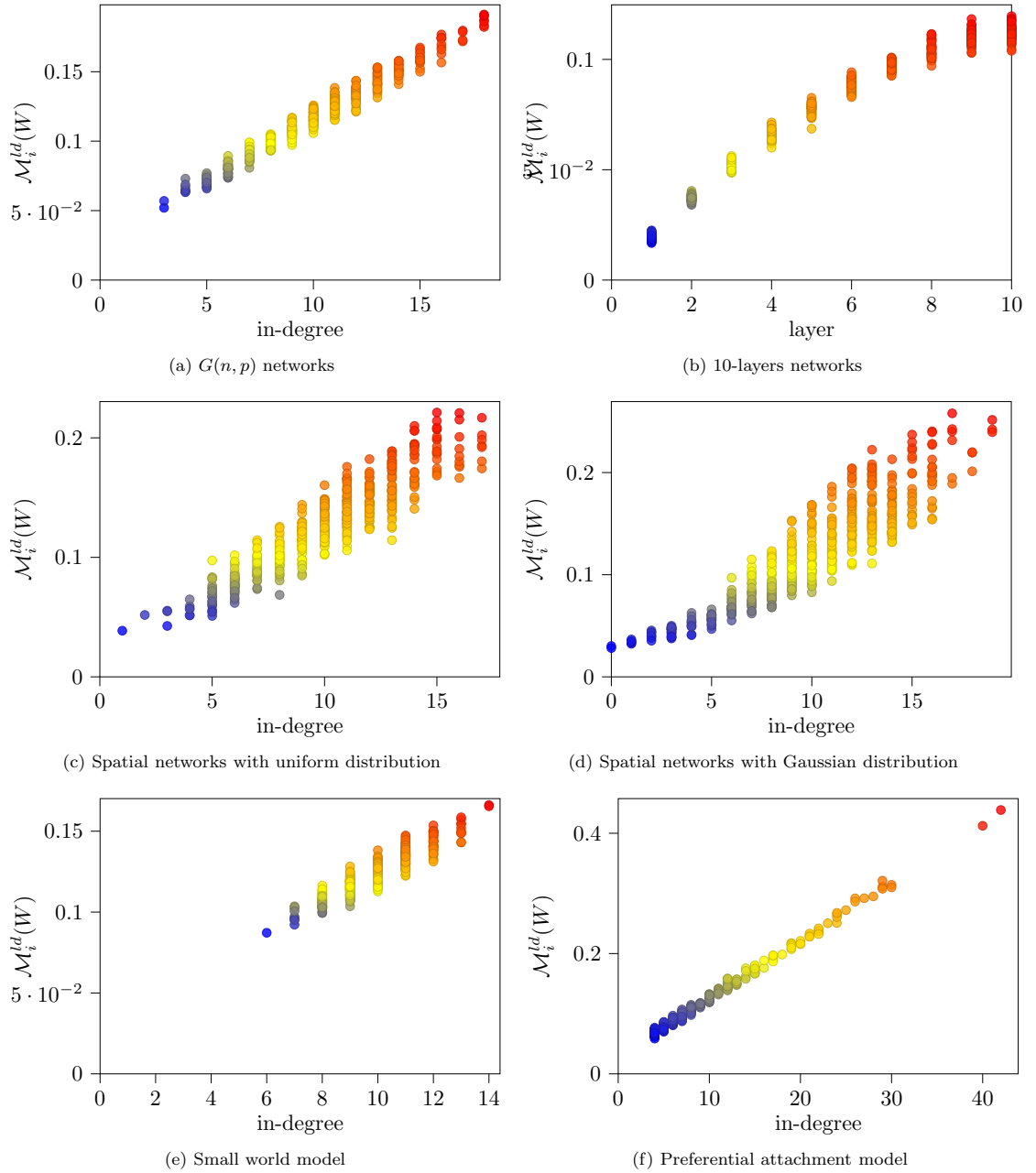


Figure 7: The correlation between the criticality and the in-degree (or the layer depth for figure (b)) in the different networks considered.

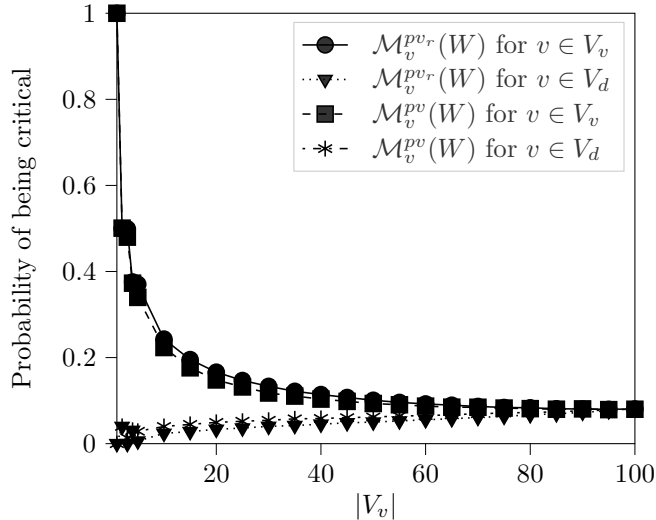


Figure 8: Probability of an agent being critical with $|V_v|$ varying from 1 to 100 in the standard proxy voting setting pv or the restricted version from Appendix A.4, pv_r . We have $|V| = 100$ and $p_d = 0.5$ in the pv setting, and W is a WVG with all weights equal to 1 and $q = 0.5$. This experiment sampled over 100,000 delegations partitions.

Comparing the two models of proxy voting In the experiments with proxy voting (PV as described in Section 4 and the restricted version PV_r described in Section A.4), we study the case when all agents have the same voting weight. Note that within either V_v or V_d that all agents have the same voting power. We performed tests on how the number of proxies affects the probability of agents being critical. We study the setting where $|V| = 100$ and $|V_v| \in [1, 2, 3, 4, 5, 10, \dots, 95, 100]$ (thus, $|V_d| = |V| - |V_v|$), the voting rule corresponds to the majority rule ($q = 0.5$), and for the PV setting we have that the probability of delegating is $p_d = 0.5$. We see the results of this numerical test in Figure 8. We see similar trends between the PV_r and PV settings, with the main difference in behaviour being determined by whether an agent is a delegator or a delegatee. Yet, the difference in probability of being critical for all voters is slightly less in the PV setting than in the PV_r setting. Returning to the comparisons between the criticality of the delegators and the delegatees, we see that when there are very few proxies in V_v , then their likelihood of being critical is very high and approaches 1. Therefore, their probability of being critical is high due to them receiving many delegations given that $p_d = 0.5$ and the choice of delegatee is small. As $|V_v|$ increases, so does the choice of delegates for those in V_d ; hence, the probability of being critical for those in V_v decreases. Conversely, the criticality of delegators in V_d slightly raises with this increase in delegation options.