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Notions of Single-Peakedness for Incomplete Preferences

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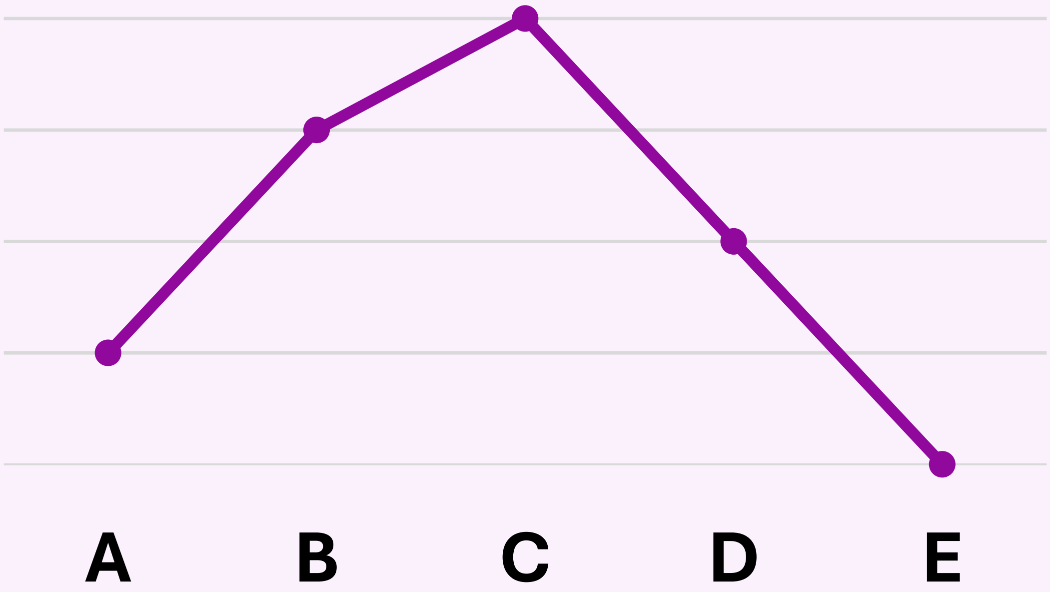
Context: Voting and preferences

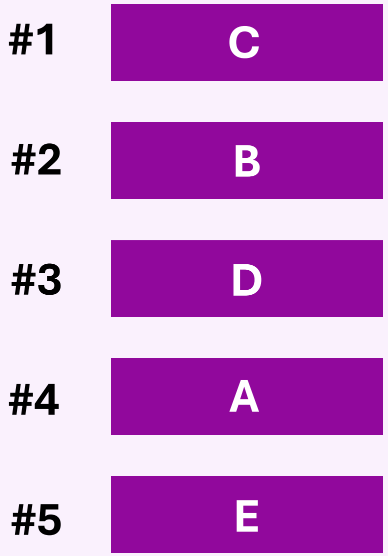
We know the preference **rankings** of several **voters** over a set of **candidates**.

	Voter 1	Voter 2	Voter 3	Voter 4
#1	C	D	A	E
#2	B	C	B	C
#3	D	E	C	D
#4	A	B	D	B
#5	E	A	E	A

What is single-peakedness?

Preferences are **single-peaked** if there is an **axis** such that for each voter, their preference decreases the further we get from the peak.





Example: preference over sauna temperature

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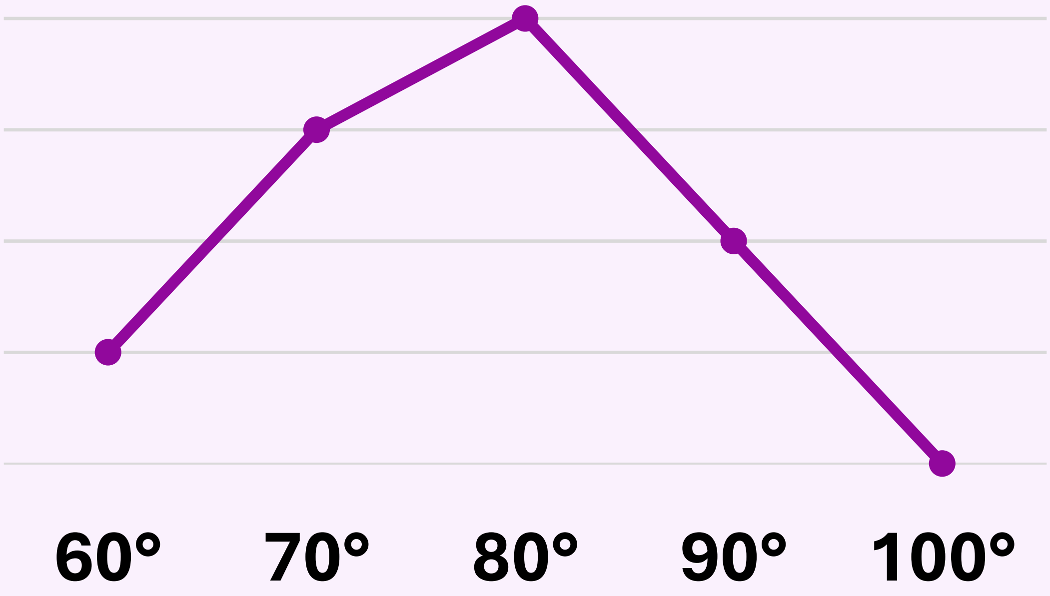
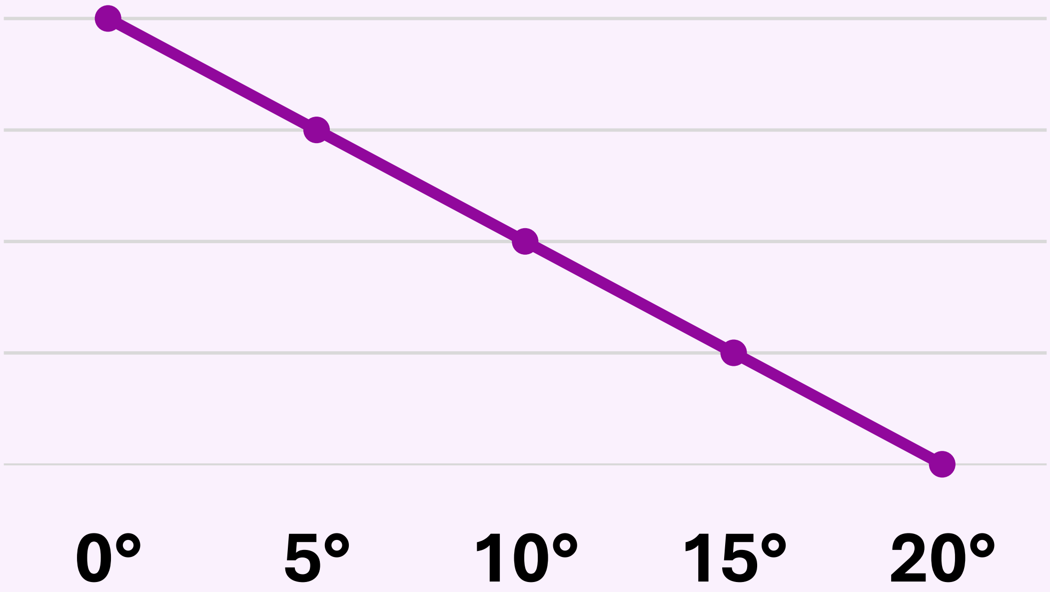


#1	80°
#2	70°
#3	90°
#4	60°
#5	100°

Example: preference over sea temperature

Preferences are **single-peaked** if there is an **axis** such that for each voter, their preference decreases the further we get from the peak.



#1	0°
#2	5°
#3	10°
#4	15°
#5	20°

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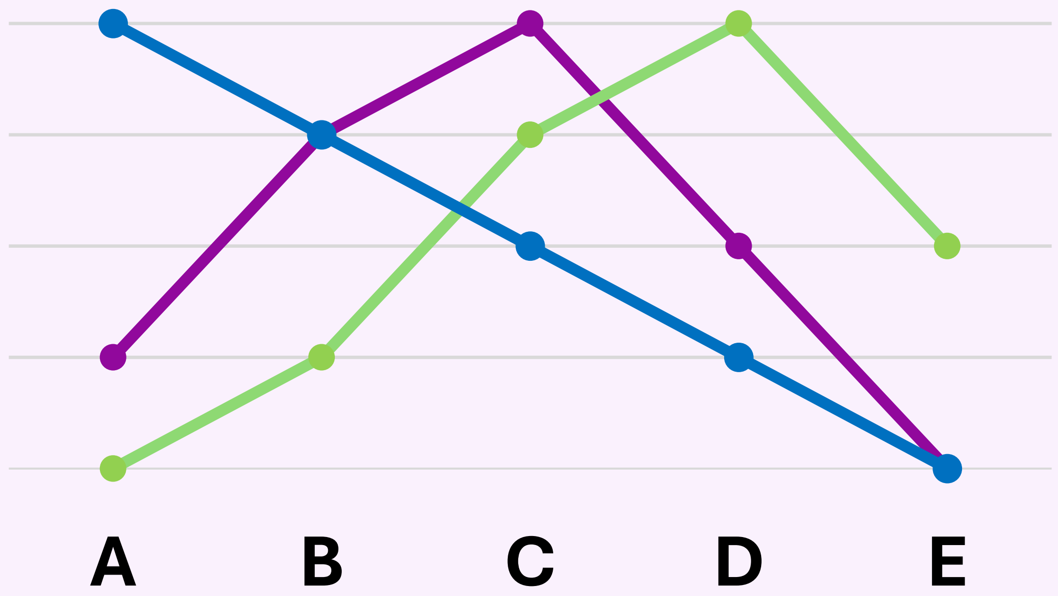

#1	C	D	A
#2	B	C	B
#3	D	E	C
#4	A	B	D
#5	E	A	E

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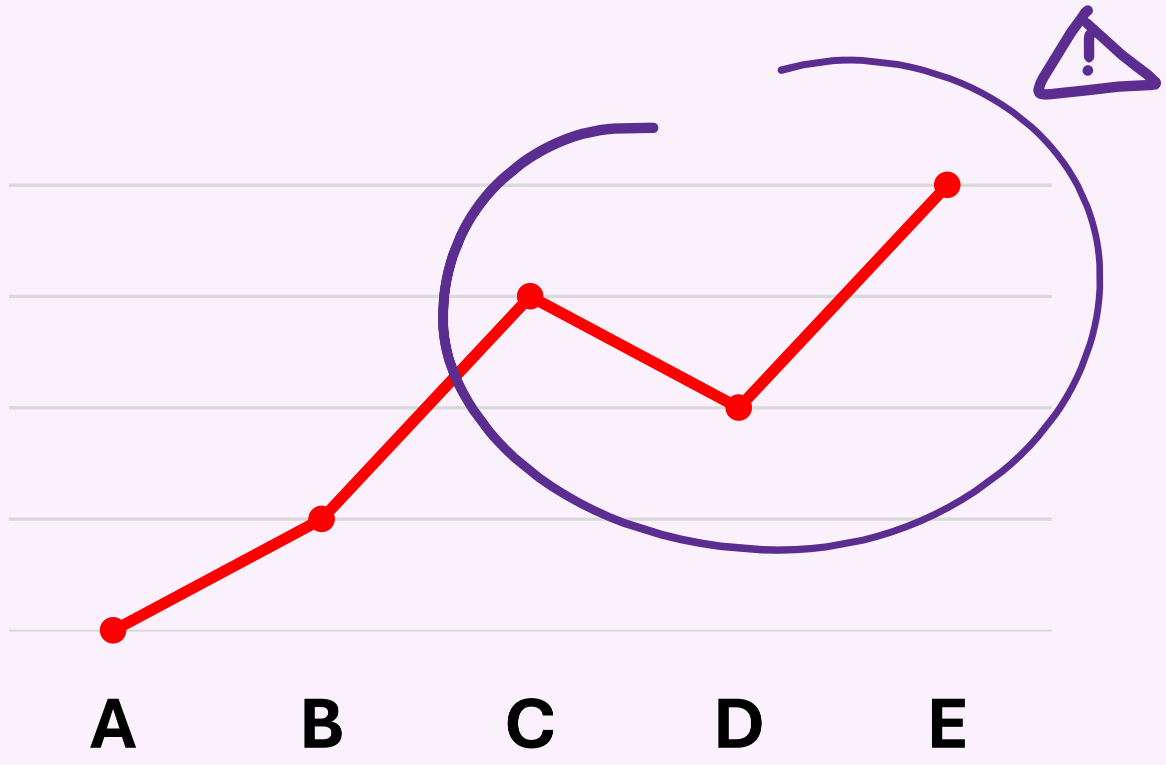
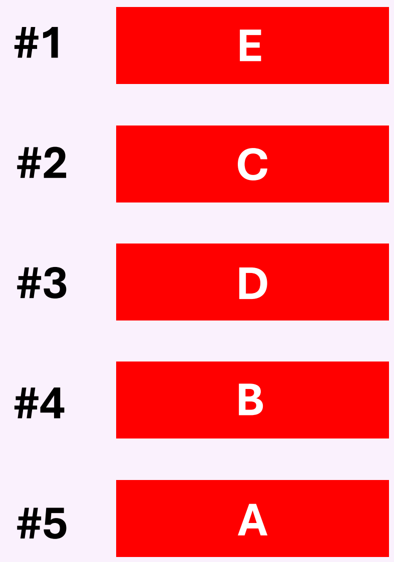




What is nice about single-peaked preferences?

1 Preference analysis

It highlights a ***meta-consensus*** on a unifying dimension for the candidates.

2 Decision making

Impossibility theorems do not apply: **there is a Condorcet winner** and the pairwise majority relation is transitive.

Single-peakedness in deliberative polls

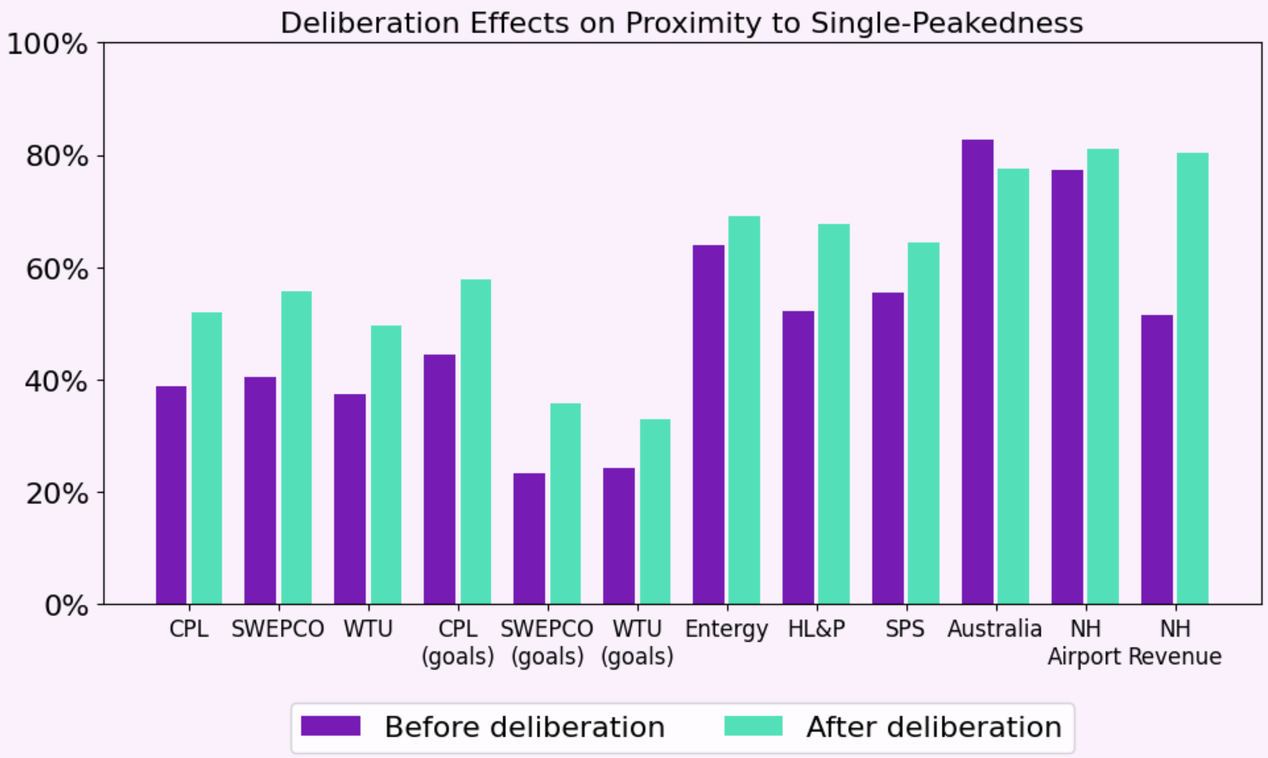
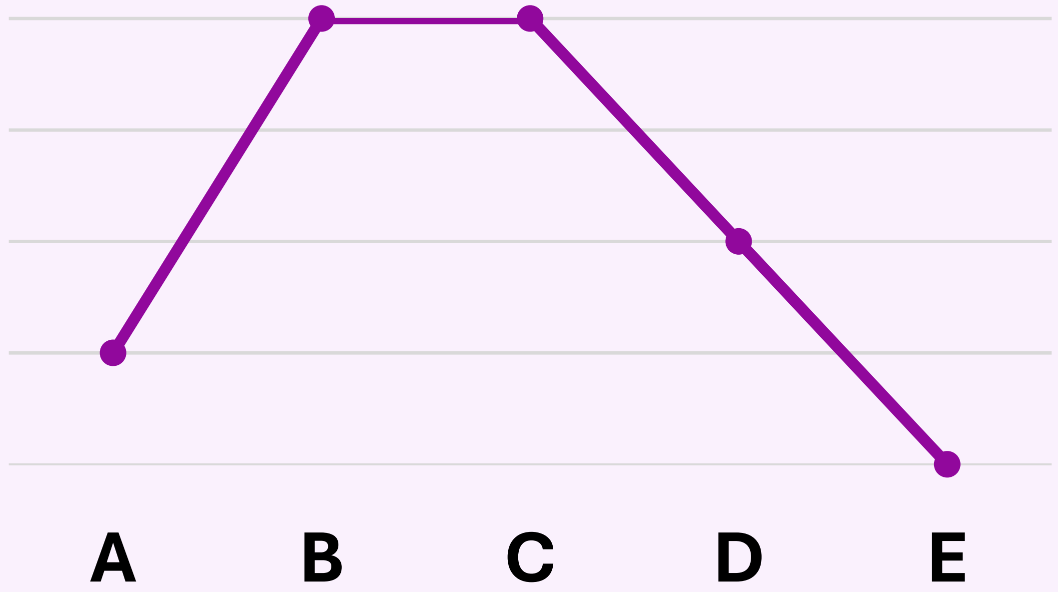


Fig. People in a Deliberative Polls
(Source: Deliberative Democracy Lab)

Deliberation, Single-Peakedness, and the Possibility of Meaningful Democracy: Evidence from Deliberative Polls. List et al. **The Journal of Politics (2013)**

What if we have indifferences?

Sometimes voters (can) express **indifferences** between some candidates in their ranking, putting them at the same rank.



#1	B=C
#2	D
#3	A
#4	E

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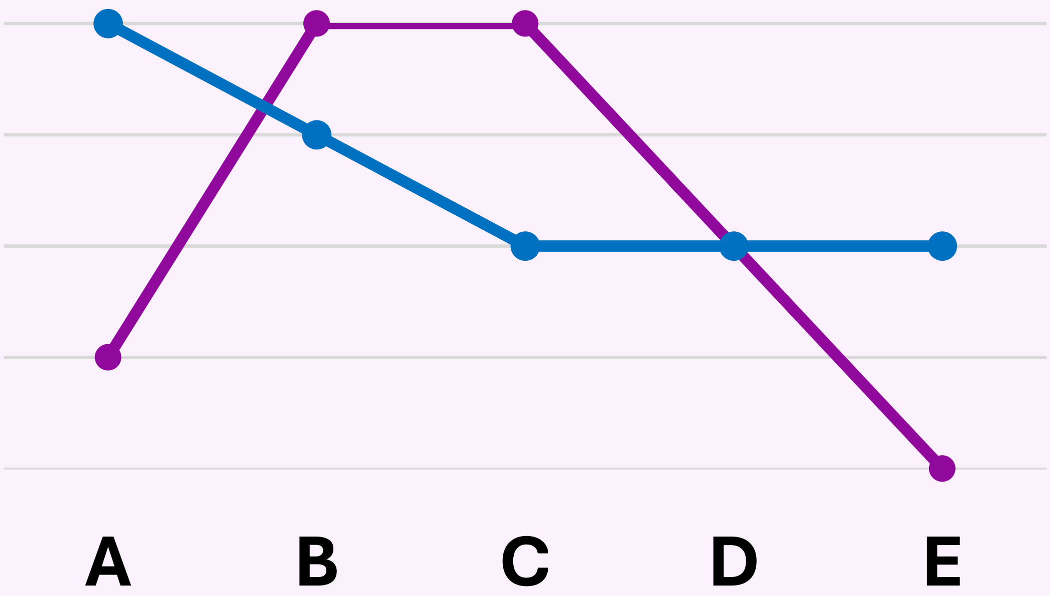
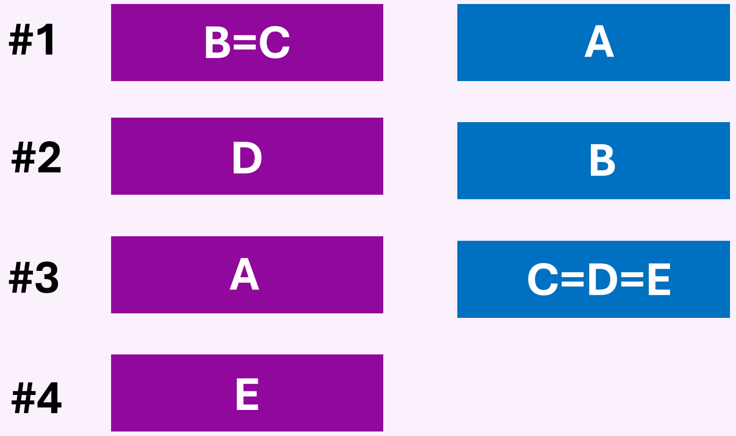
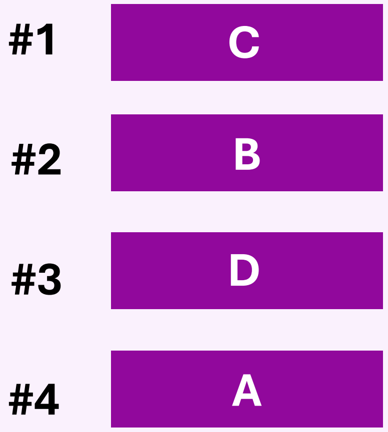




***Question:* Are these preferences single-peaked?**

What if we have incomplete preferences?

Sometimes voters (can) provide **incomplete** preferences, meaning we do not know their preferences between some candidates.



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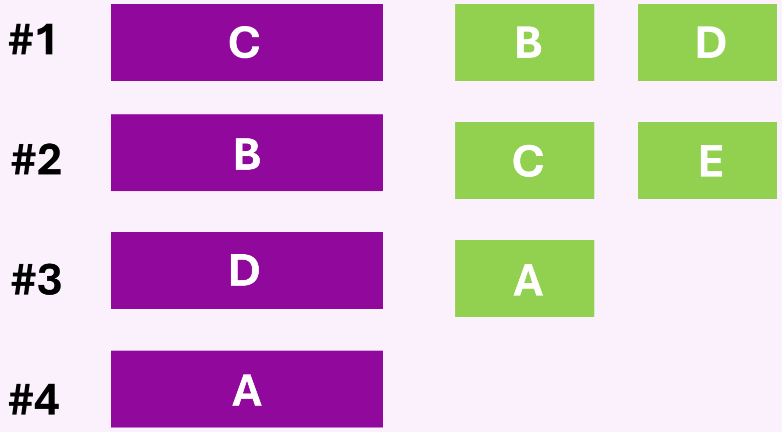

***Question:** Are these preferences single-peaked?*

Question: when preferences contain **indifferences** and **incompleteness**, how should we define single-peakedness?

- 1 -
Ten single-peakedness notions

Formal model of (weak) preferences

We have a set of n **voters** V and a set of m **candidates** C.

Voters give **preferences** over the candidates:

$A \succ_i B$ means *"voter i likes candidate* ***A more than*** *candidate* ***B****"*
$A \sim_i B$ means *"voter i likes candidate* ***A as much as*** *candidate* ***B****"*
$A \succcurlyeq_i B$ means *"voter i likes candidate* ***A at least as much as*** *candidate* ***B****"*

Example of a **full ranking**:

$$A \succ_i C \succ_i D \succ_i B \succ_i E$$

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Example of a **weak ranking**:

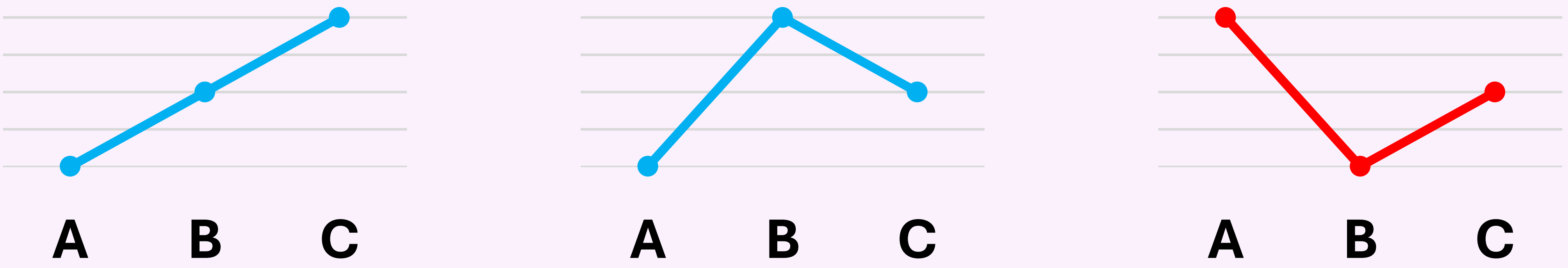
$$A \succ_i C \sim_i D \succ_i B \sim_i E$$

Formal definition of single-peakedness

With **full rankings**, the condition for single-peakedness is as follows:

If B is between A and C on the "**axis**", then:

$$B \succ_i A \text{ OR } B \succ_i C$$

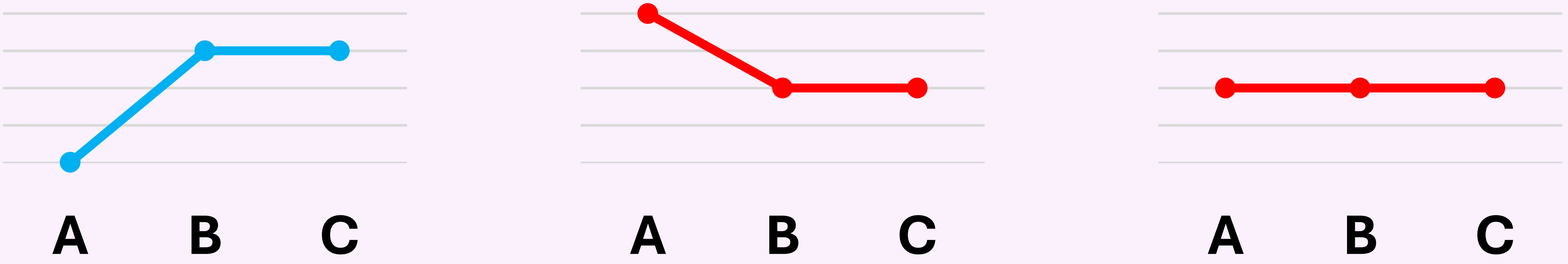


Formal definition of single-peakedness

If **B** is between **A** and **C** on the “axis”, then:

$$B \succ_i A \text{ OR } B \succ_i C$$

But if we apply it to rankings with **indifferences**:

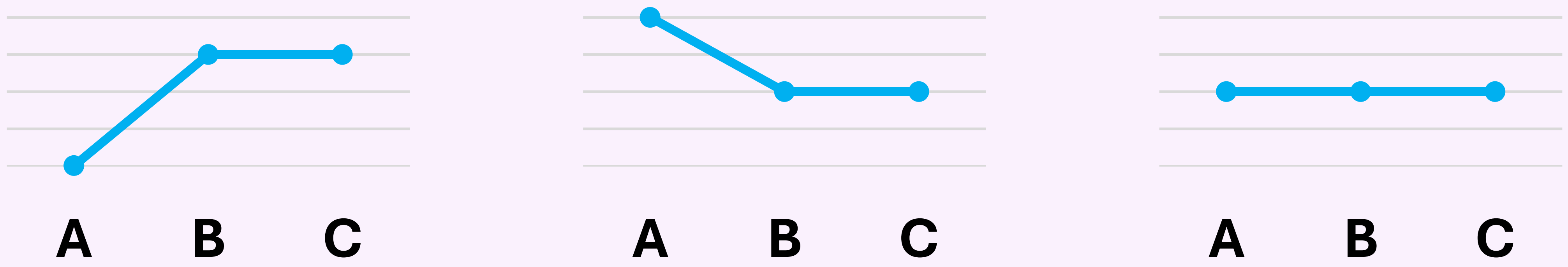


Formal definition of single-peakedness

We could slightly adapt it:

If **B** is between **A** and **C** on the “axis”, then:

$$B \succcurlyeq_i A \text{ OR } B \succcurlyeq_i C$$



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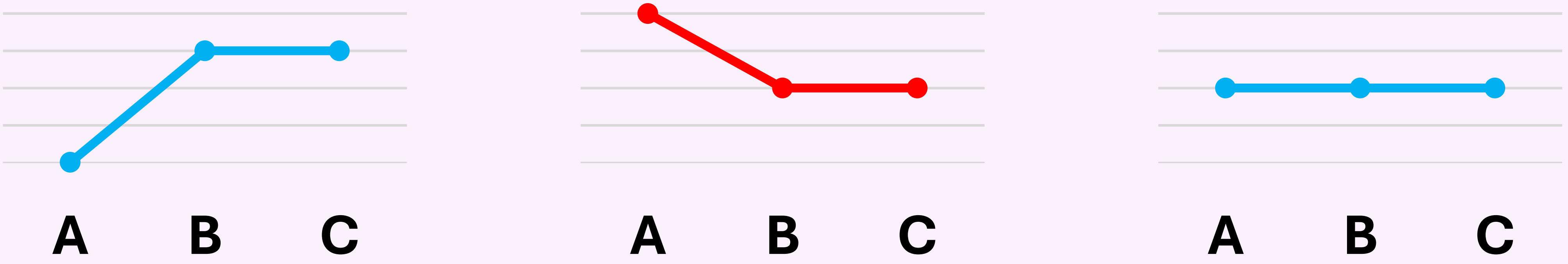


Let’s now add **incomparability**

	$\succ$	$\sim$	$?$
$B \succ_i A$ means *“voter i likes candidate* ***B more than*** *candidate* ***A****”*	✓		
$B \succcurlyeq_i A$ means *“voter i likes candidate* ***B at least as much as*** *candidate* ***A****”*	✓	✓	
$\neg(A \succcurlyeq_i B)$ means *“voter i does* ***not*** *like candidate* ***A at least as much as*** *candidate* ***B****”*	✓		✓
$\neg(A \succ_i B)$ means *“voter i does* ***not*** *prefer candidate* ***A more than*** *candidate* ***B****”*	✓	✓	✓

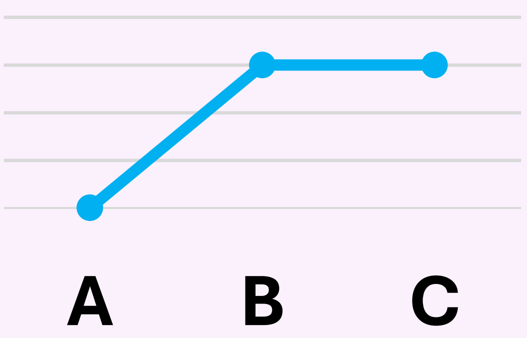
All combinations

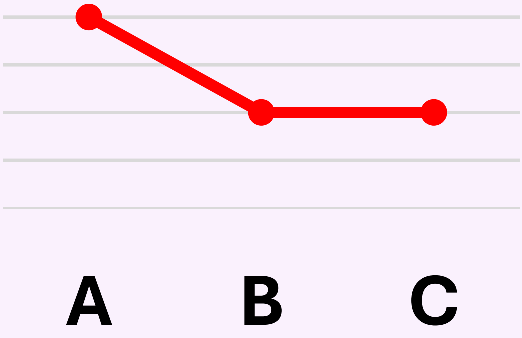
$$\begin{array}{c} B \succ_i A \\ B \succcurlyeq_i A \\ \neg(A \succcurlyeq_i B) \\ \neg(A \succ_i B) \end{array} \times \begin{array}{c} B \succ_i C \\ B \succcurlyeq_i C \\ \neg(C \succcurlyeq_i B) \\ \neg(C \succ_i B) \end{array}$$

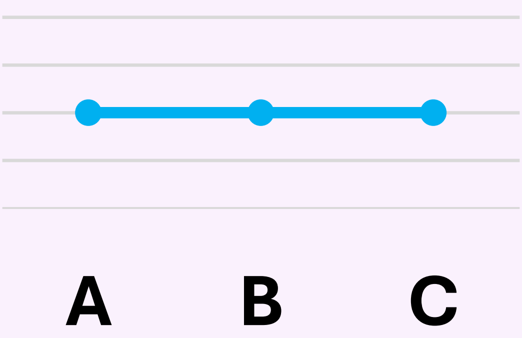
All combinations

$$B \succ_i A \qquad\qquad B \succ_i C$$
$$B \succcurlyeq_i A \qquad \times \qquad B \succcurlyeq_i C$$
$$\neg(A \succcurlyeq_i B) \qquad\qquad \neg(C \succcurlyeq_i B)$$
$$\neg(A \succ_i B) \qquad\qquad \neg(C \succ_i B)$$

$$B \succcurlyeq_i A \text{ OR } B \succ_i C$$







A hierarchy of ten notions

$B \succ_i A$ OR $B \succ_i C$

$B \succ_i A$ OR $B \succcurlyeq_i C$

$B \succ_i A$ OR $\neg(C \succcurlyeq_i B)$

$B \succcurlyeq_i A$ OR $\neg(C \succcurlyeq_i B)$

$B \succcurlyeq_i A$ OR $B \succcurlyeq_i C$

$\neg(A \succcurlyeq_i B)$ OR $\neg(C \succcurlyeq_i B)$

$B \succ_i A$ OR $\neg(C \succ_i B)$

$B \succcurlyeq_i A$ OR $\neg(C \succ_i B)$

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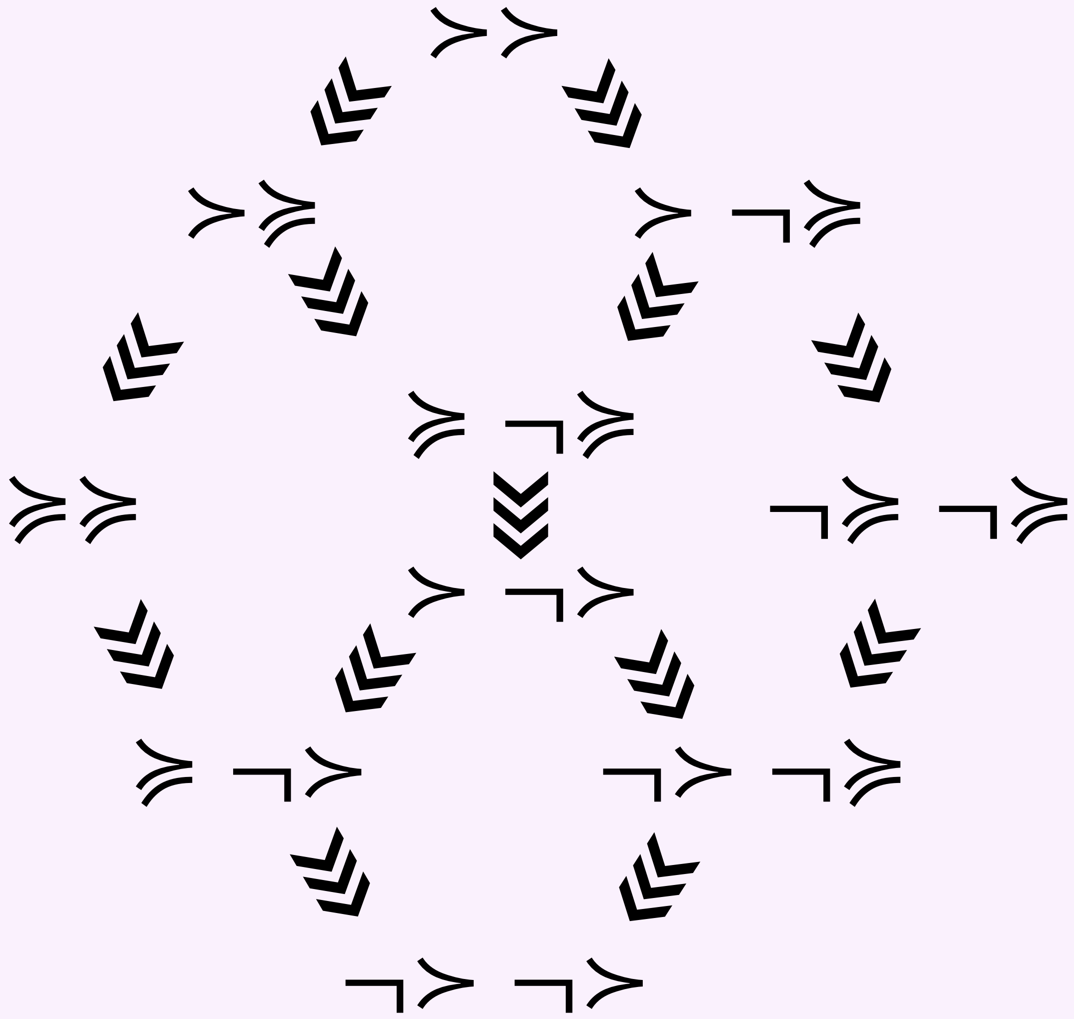

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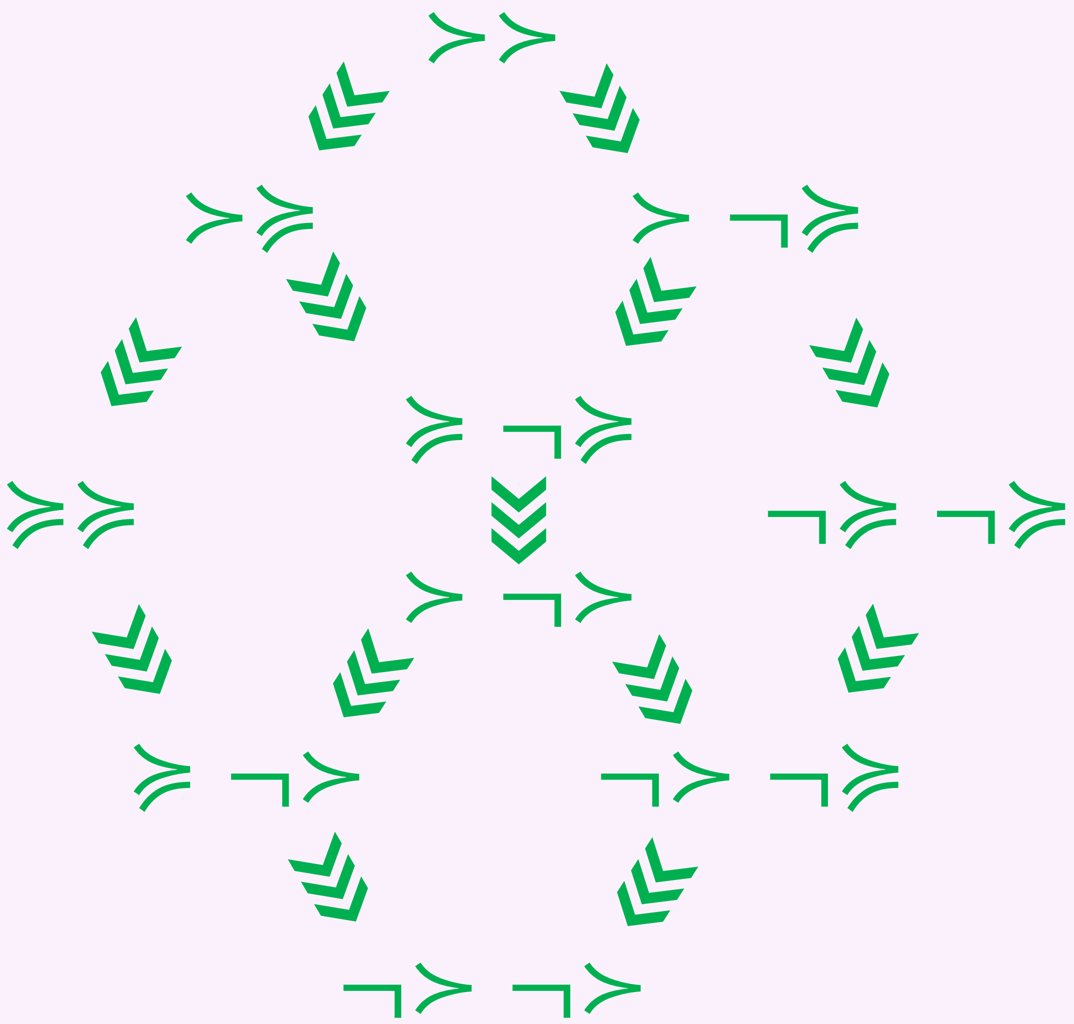
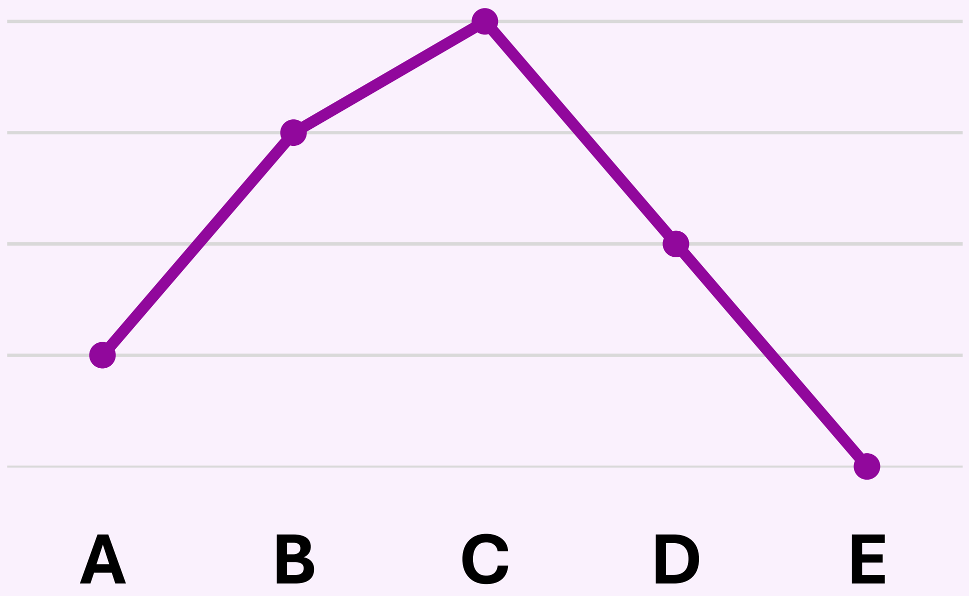


Is this single-peaked?



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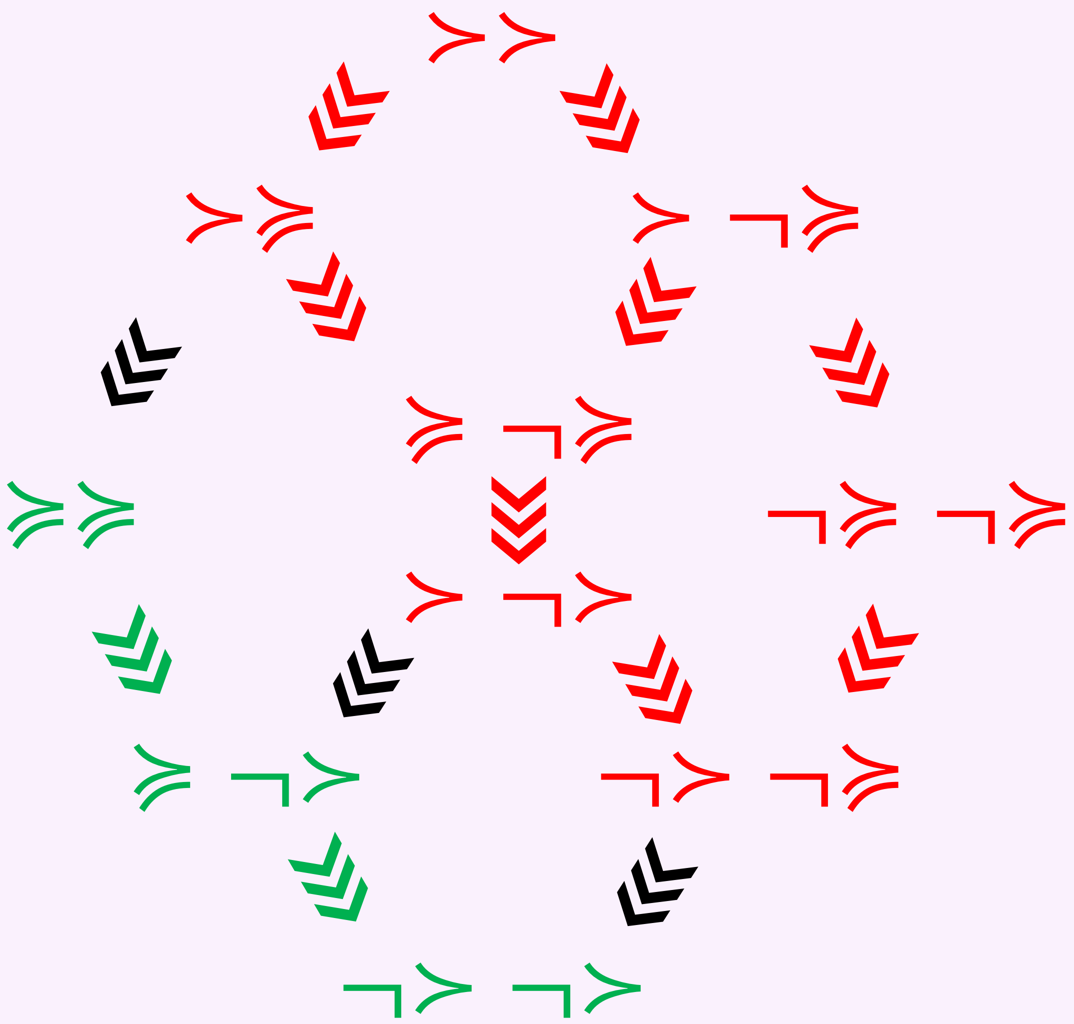

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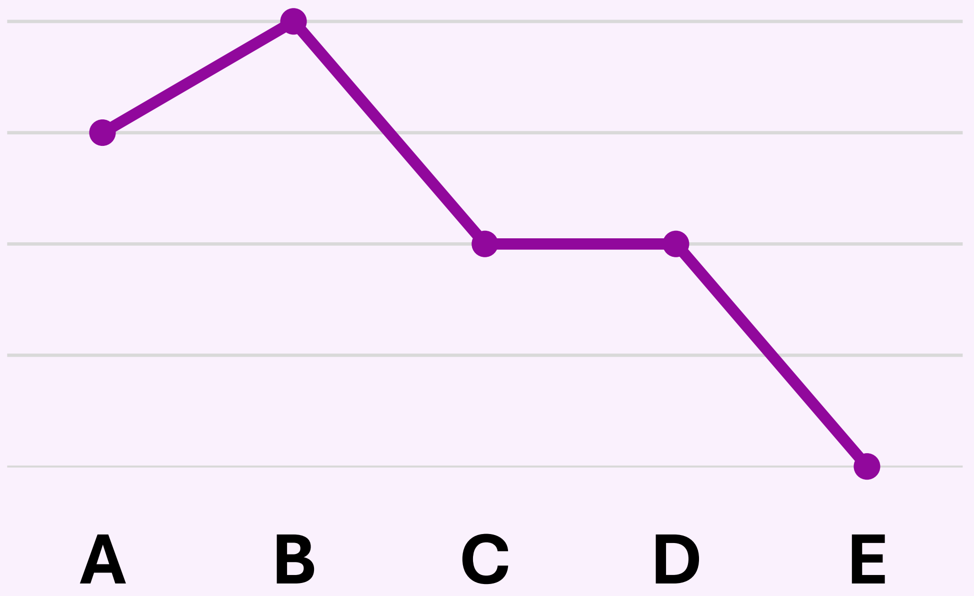
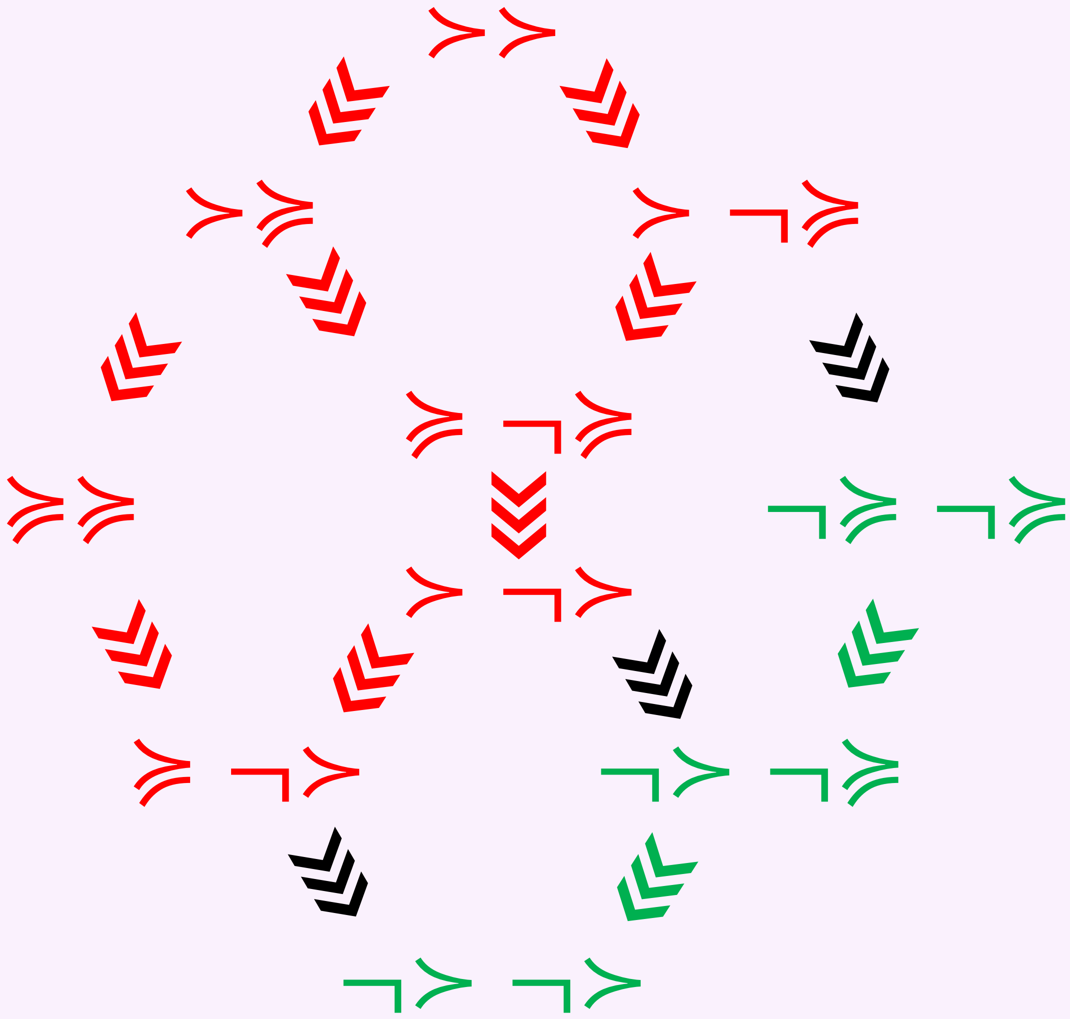
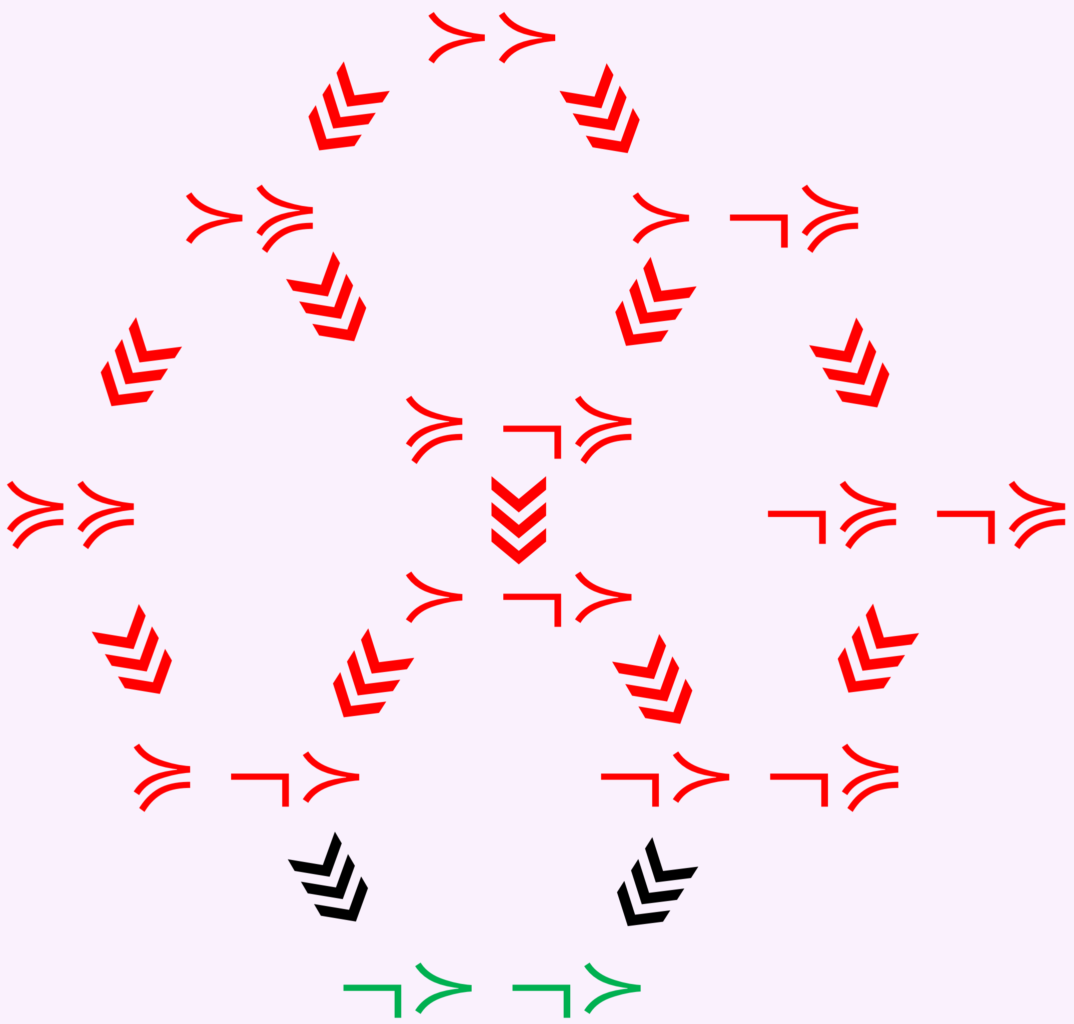


A hierarchy of **ten notions**



Is this single-peaked?

A hierarchy of ten notions



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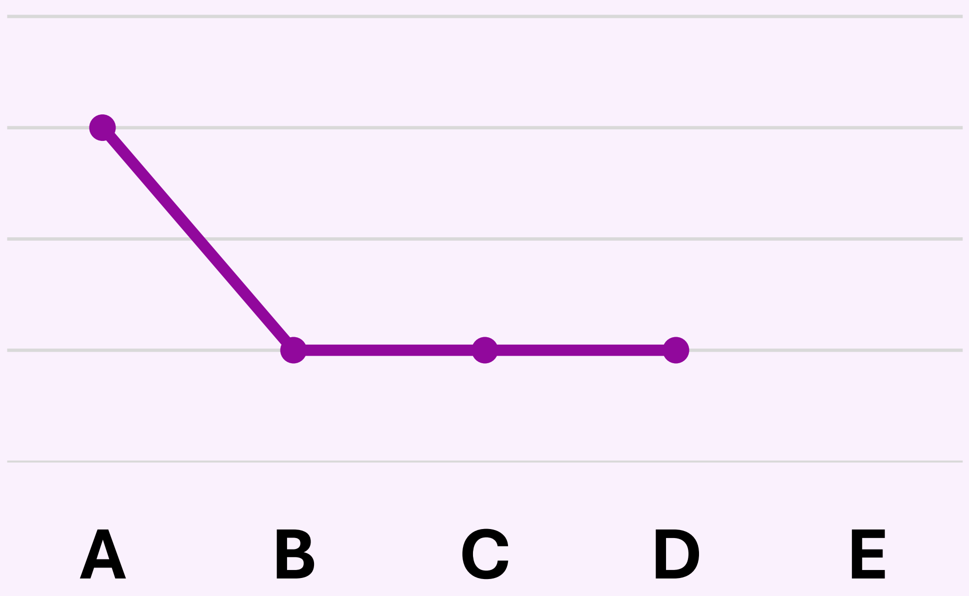


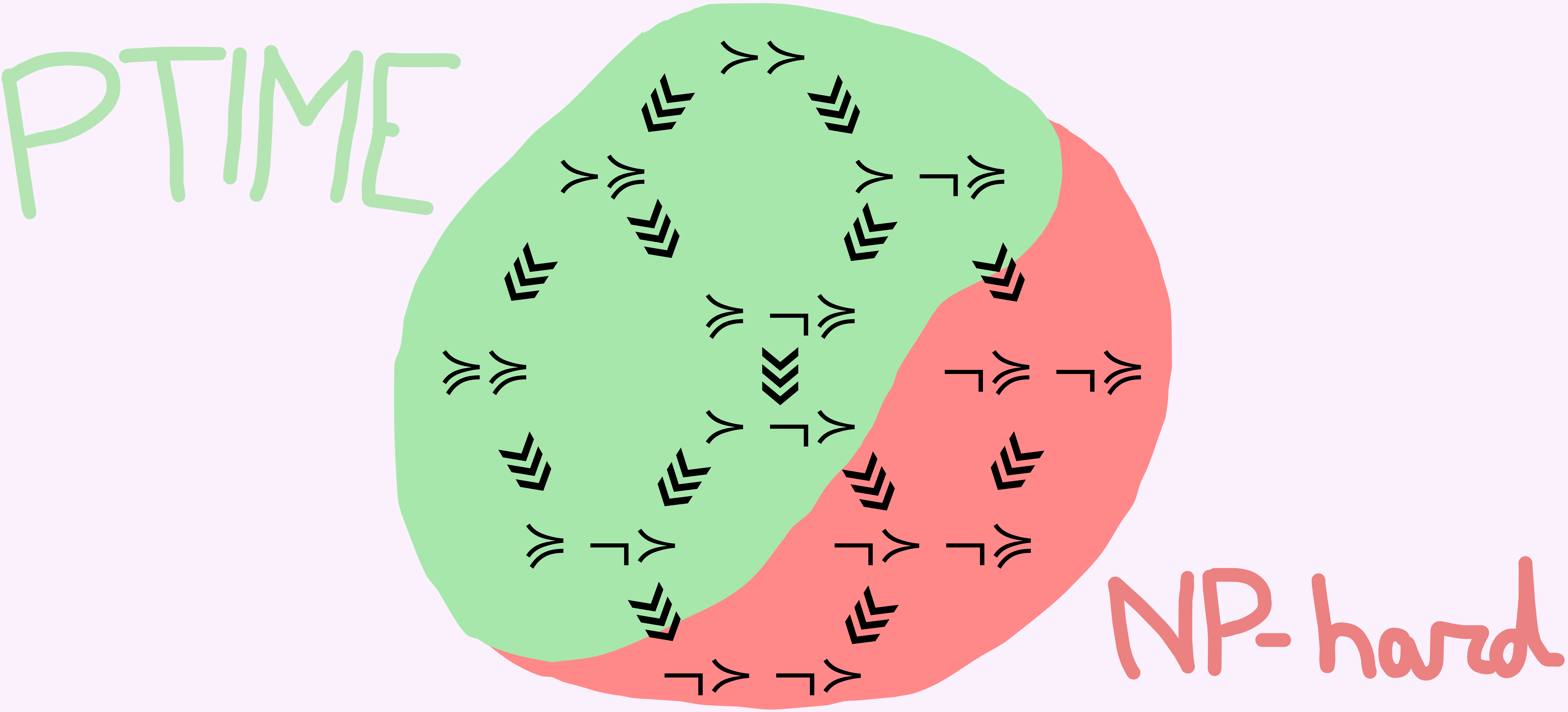
- 2 -
Theoretical Analysis of the Different Notions

Complexity of verifying single-peakedness

Question: how complex is it to compute whether a profile of preferences satisfies the single-peakedness condition?

- **PTIME:** We **can** compute it easily, which makes it practical to use.
- **NP-Hard:** We **cannot** compute easily when the number of candidates is large, making it unpractical.

Complexity of verifying single-peakedness



Transitivity of the majority relation

With full rankings, the **majority relation** is defined as follow:

$\boldsymbol{A} \succ_M \boldsymbol{B}$ if there are more voters that prefer **A** to **B** than voters that prefer **B** to **A**.

If there is a candidate **A** such that $\boldsymbol{A} \succ_M \boldsymbol{B}$ for every other candidate **B**, it is a **Condorcet** winner.

Transitivity of the majority relation

In general, there might not be a Condorcet winner, and the majority relation might be intransitive: this is the **Condorcet paradox**.

$$A \succ_1 B \succ_1 C$$
$$B \succ_2 C \succ_2 A$$
$$C \succ_3 A \succ_3 B$$

$\longrightarrow$

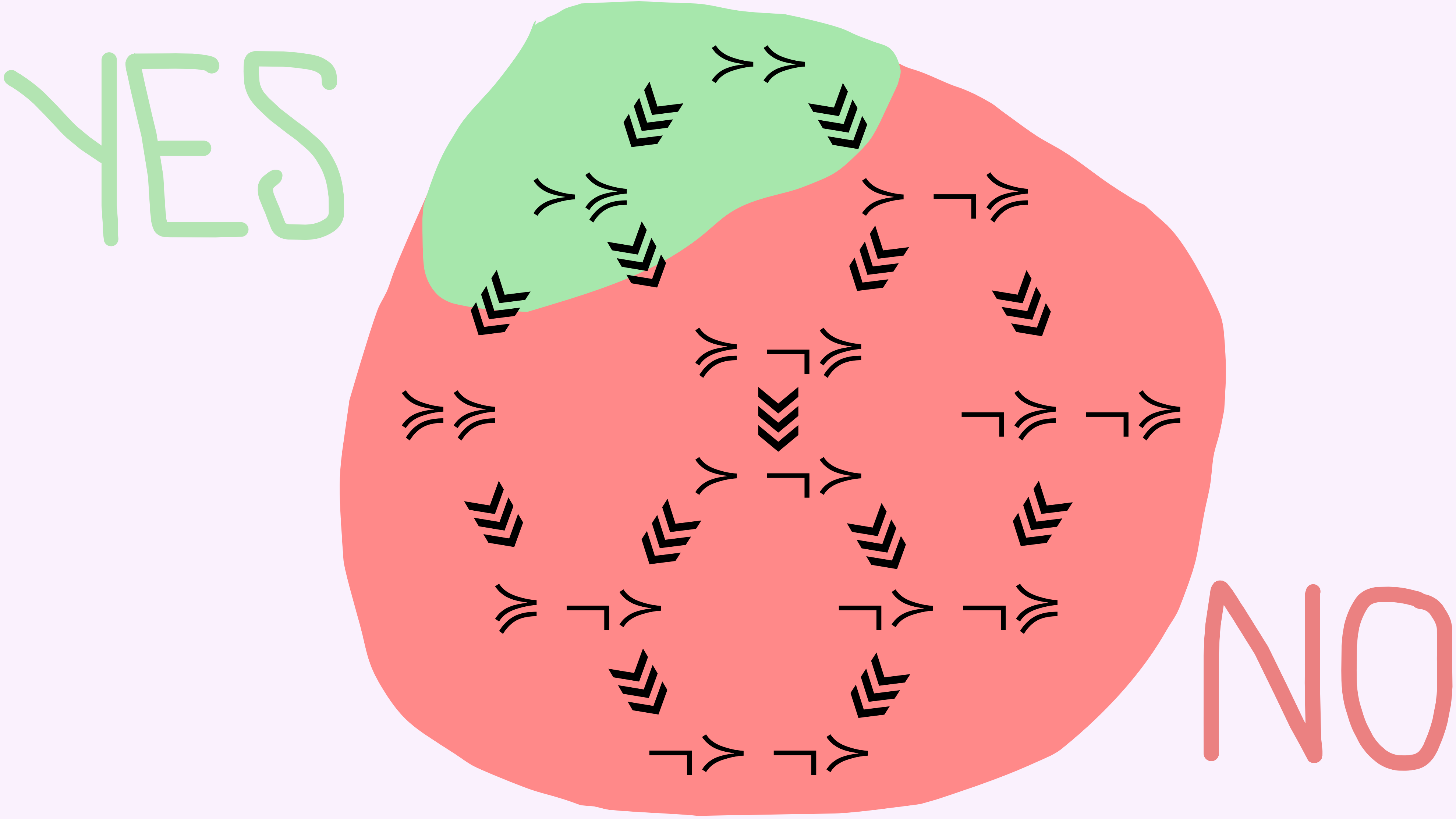
$$A \succ_M B$$
$$B \succ_M C$$
$$C \succ_M A$$

Transitivity of the majority relation

In the full rankings model, if preferences are single-peaked, **there is always a Condorcet winner and the majority relation is transitive** [Black, 1948]

Question: in the model with indifferences and incompleteness, for which notions of single-peakedness does this remain true?

Transitivity of the majority relation



- 3 -
What Difference in Practice?

Effect of deliberative polls

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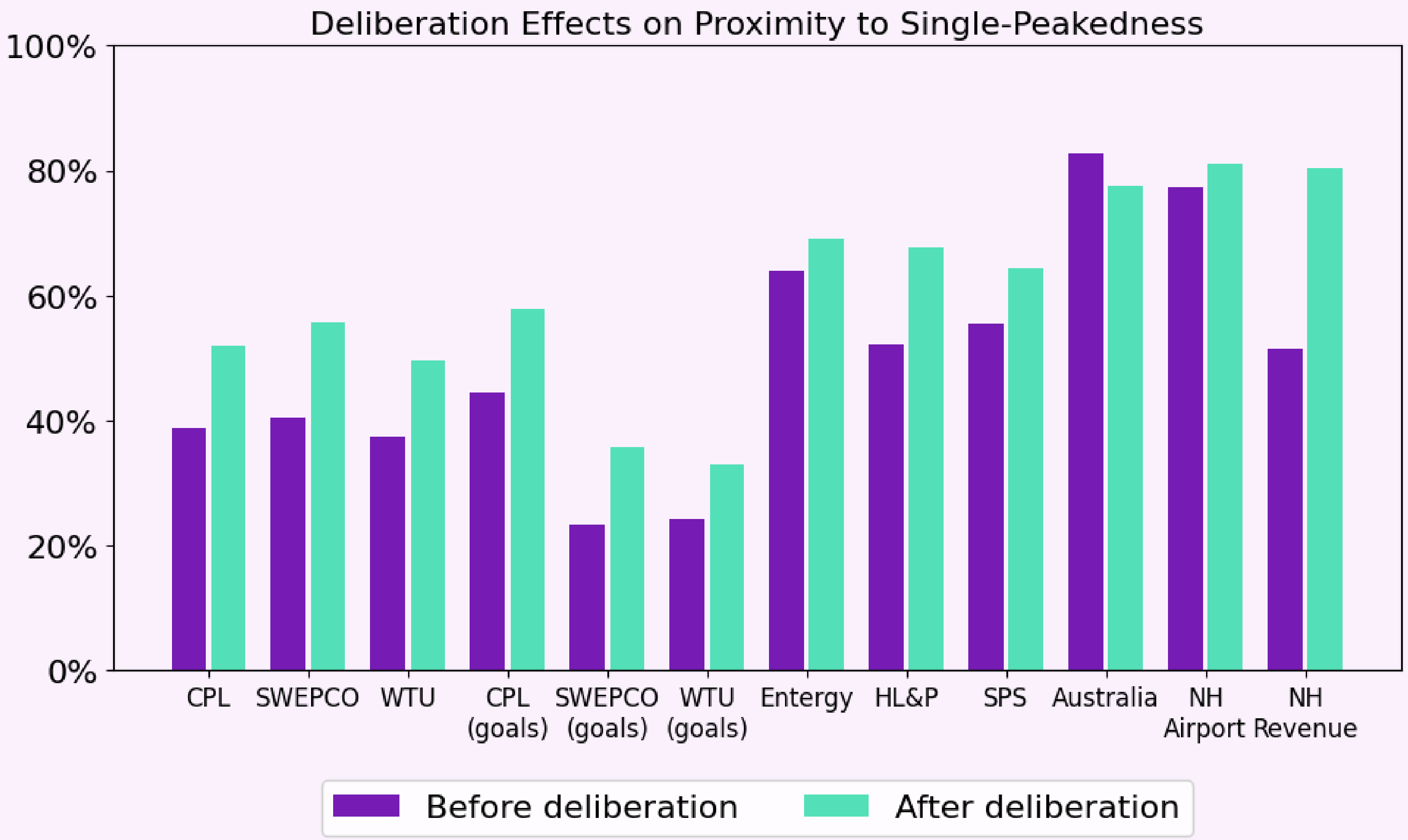
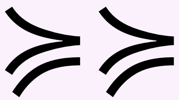
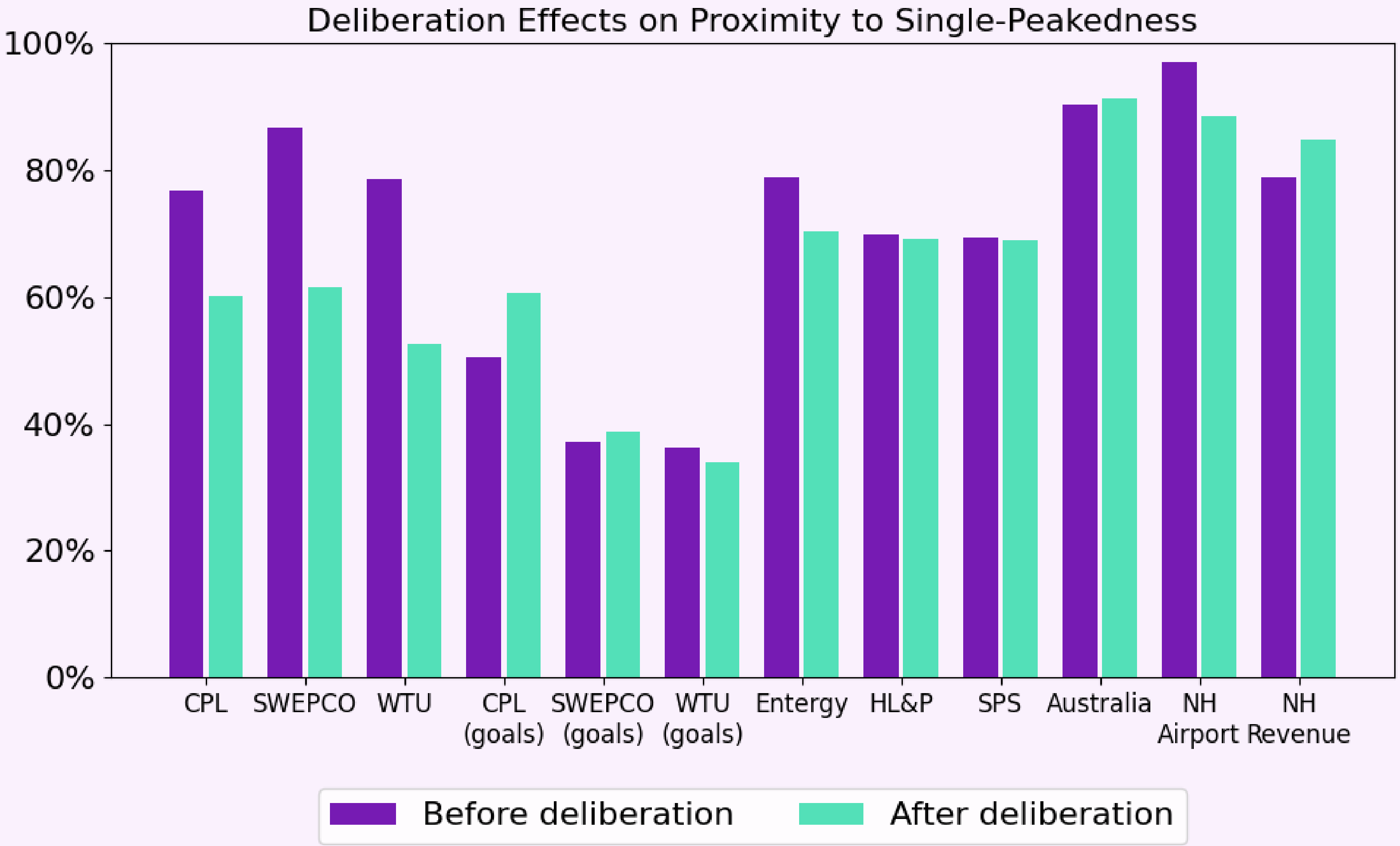
Increase in single-peakedness!



Effect of deliberative polls

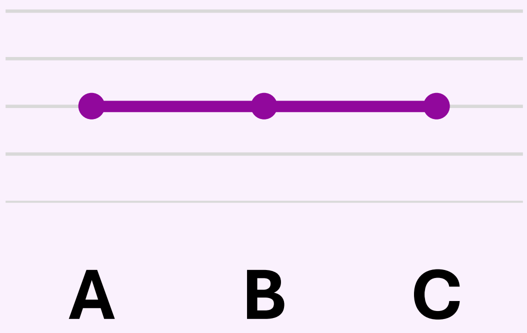


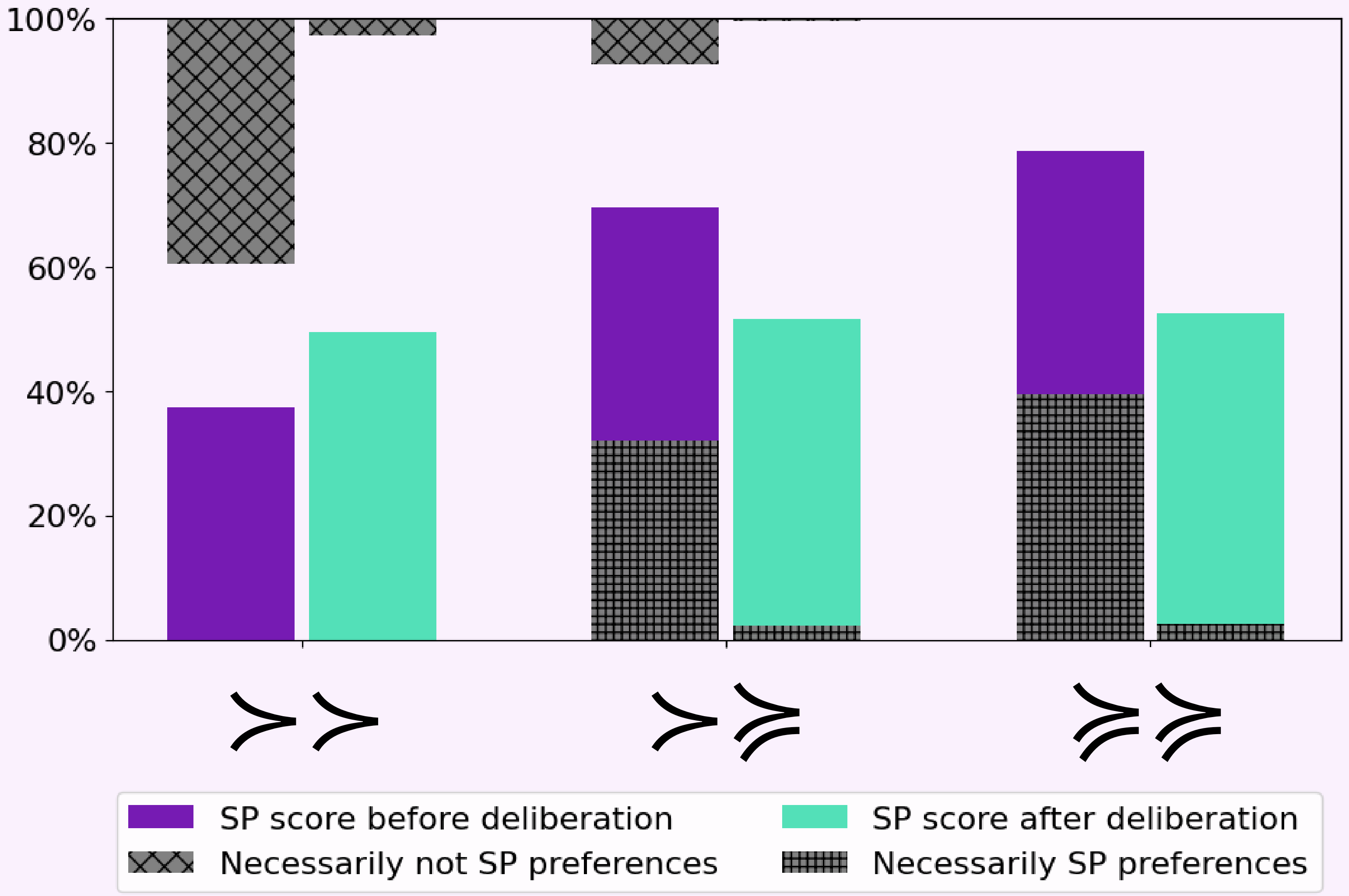
Increase in single-peakedness?



Why so much difference?

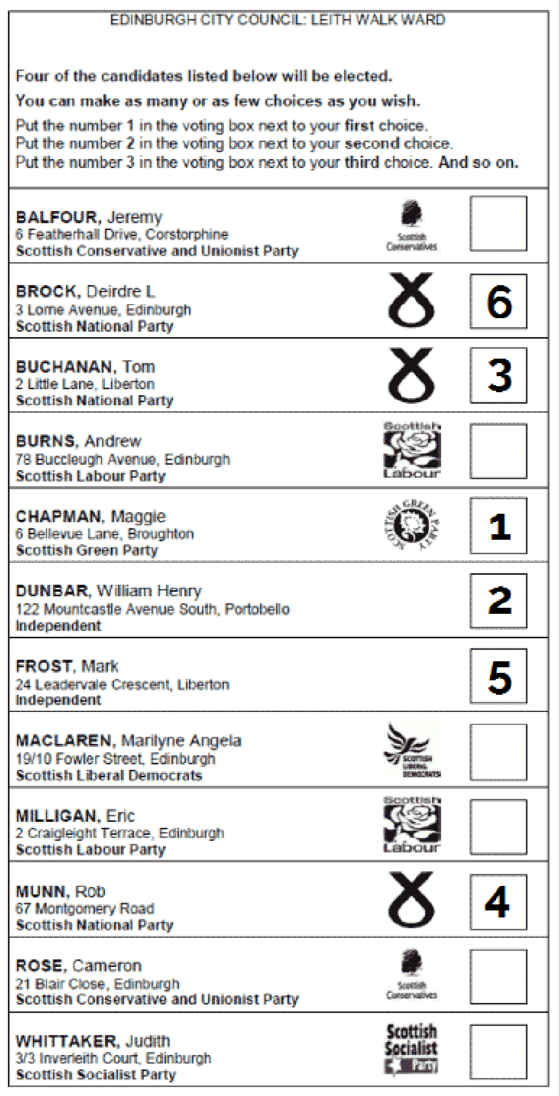
Empty preferences are **never single-peaked** for ≻≻ and are **always single-peaked** for ≽≽





Elections in **Scotland**

For local elections in Scotland, voters **can give (truncated) rankings** of candidates.



EDINBURGH CITY COUNCIL: LEITH WALK WARD

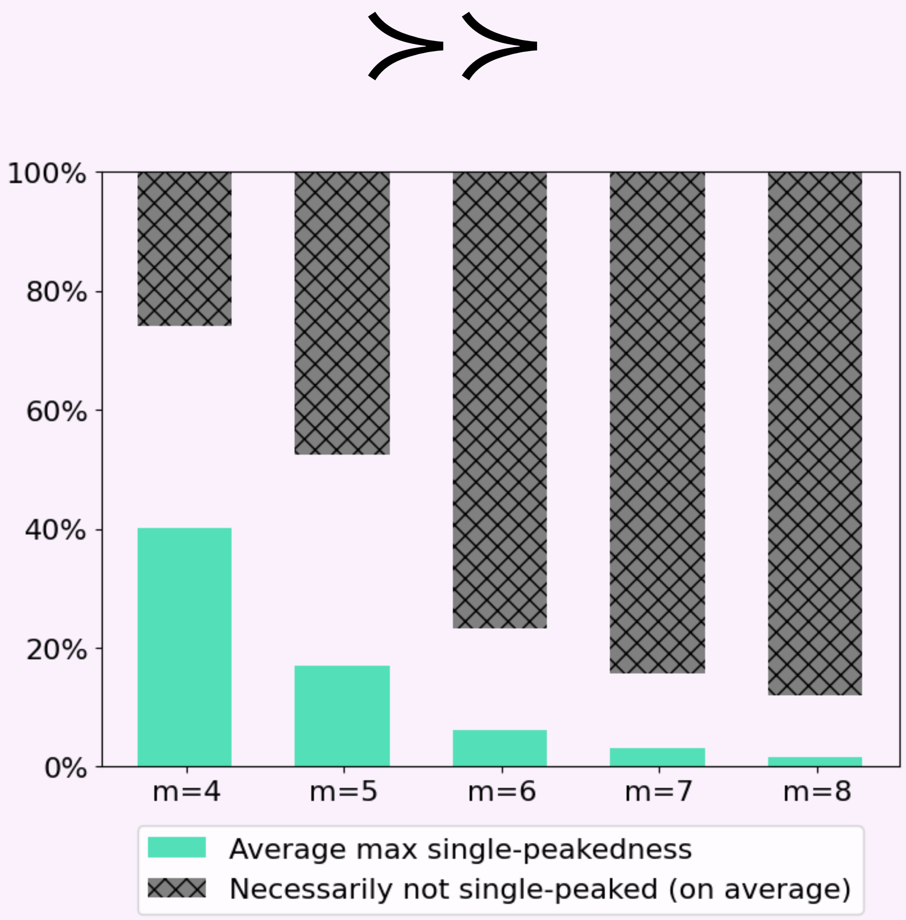
Four of the candidates listed below will be elected.
You can make as many or as few choices as you wish.

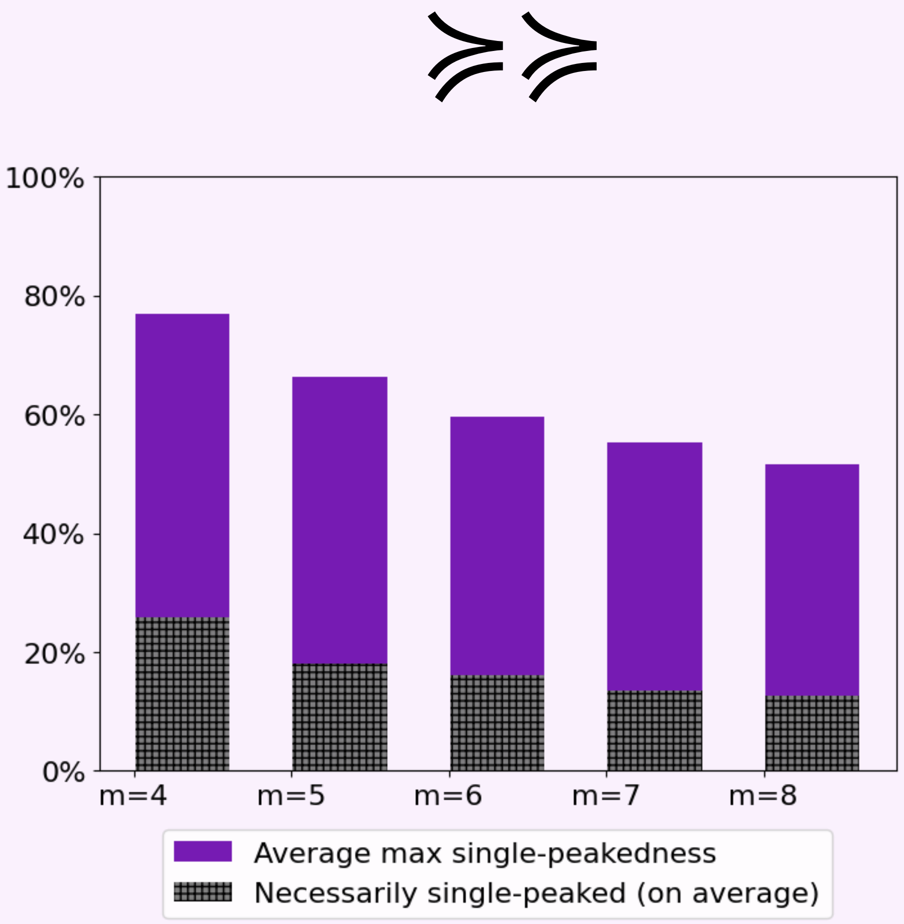
Put the number 1 in the voting box next to your **first** choice.
Put the number 2 in the voting box next to your **second** choice.
Put the number 3 in the voting box next to your **third** choice. And so on.

BALFOUR, Jeremy 6 Featherhall Drive, Corstorphine Scottish Conservative and Unionist Party	
BROCK, Deirdre L 3 Lorne Avenue, Edinburgh Scottish National Party	6
BUCHANAN, Tom 2 Little Lane, Liberton Scottish National Party	3
BURNS, Andrew 78 Buccleugh Avenue, Edinburgh Scottish Labour Party	
CHAPMAN, Maggie 6 Bellevue Lane, Broughton Scottish Green Party	1
DUNBAR, William Henry 122 Mountcastle Avenue South, Portobello Independent	2
FROST, Mark 24 Leadervale Crescent, Liberton Independent	5
MACLAREN, Marilyne Angela 19/10 Fowler Street, Edinburgh Scottish Liberal Democrats	
MILLIGAN, Eric 2 Craigleight Terrace, Edinburgh Scottish Labour Party	
MUNN, Rob 67 Montgomery Road Scottish National Party	4
ROSE, Cameron 21 Blair Close, Edinburgh Scottish Conservative and Unionist Party	
WHITTAKER, Judith 3/3 Inverleith Court, Edinburgh Scottish Socialist Party	

Fig. Ballot in Scottish election

Elections in Scotland





- 4 -
Further Work and Possibility for Collaboration

Summary and Take away

When preferences contain indifferences and/or incomparability, **there are several way to define single-peakedness.**

In theory, only some of these notions **keep the nice properties** of the classical single-peakedness.

In practice, some notions are very sensitive to incomplete preferences. One **must be careful with drawing conclusions** when using only one notion.

Further **work**

Comparing the notions on **more real-world data**, in particular data with both indifferences ***and*** incompatibilities (maybe by running our own experiment).

Finding **normative arguments** to distinguish the different notions.

What features of single-peakedness do you think are the most important?